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Knowing What Counts

Irish Primary Teachers' Mathematical Knowledge for Teaching

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This report draws substantially on my Ph.D. research which was supervised by Professor Deborah Ball at the University of Michigan and completed in May 2008. Readers who are interested in more detailed or technical information about the study are invited to read the dissertation, which is available from me on request.

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List of Abbreviations

CCK	Common Content Knowledge
CM	Centimetre
DES	Department of Education and Science
IRT	Item Response Theory
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
MKT	Mathematical Knowledge for Teaching
NCCA	National Council for Curriculum and Assessment
OECD	Organisation for Economic Co-operation and Development
SCK	Specialised Content Knowledge

Summary of Findings and Recommendations

Policymakers and educators in several countries around the world are interested in learning more about teachers' mathematical knowledge as a means of raising pupils' achievement in mathematics. Researchers at the University of Michigan who studied records of mathematics teaching – videotapes of lessons, teachers' notes, and pupils' work – observed that much of the work of teaching mathematics requires teachers to use a special type of mathematical knowledge. They conceptualised this kind of mathematical knowledge as "mathematical knowledge for teaching," which is often referred to by its acronym, MKT (Ball & Bass, 2003). MKT is subdivided into the domains of common content knowledge, specialised content knowledge, knowledge of content and students, and knowledge of content and teaching.

When developing the theory of MKT, the researchers studied records of Deborah Ball's third grade mathematics teaching and drew on other mathematics education research. They identified several tasks that teachers do when they teach mathematics, such as

- responding to pupils' questions
- choosing useful examples
- planning lessons
- appraising and modifying textbooks and
- assessing pupils' learning.

Ball, Bass and their colleagues[1] argue that these tasks are mathematical and that teachers need substantial mathematical knowledge to carry out the tasks. This report – based on the first national study of primary teachers' MKT anywhere in the world – identifies mathematical work done by Irish teachers. It finds that the work done by Irish teachers is largely similar to work done by teachers observed in the United States, which suggests that the knowledge requirements are similar in both settings.

In order to learn more about the knowledge requirements for teaching mathematics in the United States, hundreds of multiple-choice items to measure teachers' MKT were developed. A selection of the items were adapted for use in Ireland and used to study Irish teachers' mathematical knowledge. In order to ascertain whether the measures could be validly used in Ireland, ten teachers who responded to the measures were videotaped teaching mathematics; and the mathematical quality of their instruction (Hill et al., 2008) was related to their scores on the measures. In general, teachers who scored higher on the MKT measures exhibited instruction of a higher mathematical quality than teachers with lower scores on the measures. The multiple-choice items were then administered to 501 teachers selected from a random, representative sample of Irish schools.

The main finding of the study is that mathematical knowledge for teaching varies widely among Irish primary teachers with the highest scoring teachers responding correctly to over 60% more of the measures than the lowest scoring teachers. In addition, Irish teachers exhibited specific strengths and areas of difficulty in their responses to the items. Irish teachers' strengths included

- identifying and classifying pupils' mistakes
- representing fractions in diagrammatic form and
- responding to algebra items.

Difficulties for Irish teachers included

- applying definitions and properties of shapes
- identifying and applying properties of numbers and operations
- attending to explanations and evaluating pupils' understanding and
- linking fraction calculations to word problems.

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[1] The author is a member of the Learning Mathematics for Teaching research team. The Principal Investigators of this team are Ball, Bass and Hill. However, the research group will be referred to in the third person throughout this report.

This report recommends a set of actions that will systematically raise the mathematical knowledge for teaching of Irish teachers generally and address specific areas which Irish teachers find problematic. To achieve these goals, an individual or a committee should be appointed to implement, monitor and rigorously evaluate the initiatives below. Too often evaluation of professional development for teachers has been absent or weak. But implicit in each initiative listed below is the requirement that its impact on teacher knowledge and/or on instruction be evaluated. Instruments used in the study reported here can be used for such evaluation. Initiatives which do not raise teacher knowledge or improve instruction need to be reconsidered.

1. Raise teachers' mathematical knowledge for teaching by designing and subsequently delivering professional development that is grounded in the practice of mathematics teaching. Such professional development will be built around mathematics teaching laboratories, around Japanese-style lesson study or around video records of practice. There is an immediate need to begin building capacity for coordinating such professional development by selecting and preparing teacher leaders in mathematics around the country.

2. From a specified date, use only mathematics textbooks in Irish schools that are approved by the Department of Education and Science. Criteria for such approval should require authorship by a multi-disciplinary team of teachers, teacher educators and mathematicians with experience in textbook design and expertise in mathematical knowledge for teaching (MKT).

3. Require all prospective teachers to study mathematical knowledge for teaching (MKT) as part of their initial teacher education programmes.

4. Investigate the practicality of having specialist teachers of mathematics in some schools. For example, teachers might "swap" classes for teaching specific subjects in which they have particular expertise.

5. Use online environments, with accompanying videos of mathematics teaching, to offer courses for teachers in MKT and follow-up discussions.

6. Raise the mathematics requirement for entry to teacher education.

7. Support research that investigates the relationship between teachers' mathematical knowledge and pupil attainment. Ireland is the first country where a national study of primary teachers' mathematical knowledge for teaching has taken place. In order to build on this initiative, the following research questions should receive priority:
   a. Is there a link between teacher knowledge and pupils' attainment in Ireland?
   b. How do teachers and prospective teachers acquire mathematical knowledge for teaching?
   c. Apart from what has been learned about mathematical knowledge for teaching in the United States, what additional elements of MKT do Irish teachers know and need to know?
   d. What mathematical knowledge for teaching is used and needed by teachers of early childhood classes?
   e. What mathematical knowledge for teaching is used and needed by post-primary teachers?

01 Introducing Mathematical Knowledge for Teaching

1.1 The Importance of Mathematics

When Ireland began to envision its future as a knowledge society, its enthusiasm for promoting the learning of mathematics and science soared. Policy statements, reports and curricula emphasised the importance of success in mathematics. Policy documents such as the *Strategy for Science, Technology and Innovation: 2006–2013*[2] and *Future Requirements for High-Level ICT Skills in the ICT Sector*[3] acknowledged the importance of mathematics in a knowledge society; they made proposals to ensure that mathematically literate individuals will graduate from Irish schools, colleges and universities. Initiatives such as the National Centre for Excellence in Mathematics and Science Teaching and Learning[4] and other initiatives of the Strategic Innovation Fund[5] provide evidence of the Government's commitment to enhancing mathematics education. The primary school curriculum describes mathematics as "an essential tool for the child and adult" which "enriches [people's] understanding of the world in which they live." It further acknowledges the "profound influence" of mathematics on "the development of contemporary society" (Government of Ireland, 1999a, p. 2).

1.2 Student Achievement in Mathematics

When it comes to achievement in mathematics, however, Ireland's education system fails many pupils. High levels of failure in state exams are accompanied by unequal achievement among pupils based on their home backgrounds. In 2005 10 per cent of all pupils who sat a Leaving Certificate examination in mathematics received a failing grade. Although some failure may be attributed to pupils taking the exam at an inappropriate level, over 7 per cent of pupils failed the foundation level course and 12 per cent failed the ordinary level.[6] These levels of failure have been stubbornly persistent over the last four years, at least.[7] In international tests Irish pupils' performance in mathematics has been average, which is disappointing when compared to their scores in science and literacy where pupils exceed the average (e.g. Eivers, Shiel, & Cunningham, 2007). At primary school level, one study found that only a handful of pupils attending schools designated as disadvantaged achieved above the 80th percentile on national standardised mathematics tests and about two-thirds of such pupils scored at or below the 20th percentile (Department of Education and Science, 2005b). Such inequalities are consistent with other research (Weir, Milis, & Ryan, 2002). In addition to low and unequal mathematical achievement, concern has been expressed about the nature of pupils' mathematical knowledge.

The Chief Examiner's 2005 report on pupil performance in Leaving Certificate mathematics found that many pupils demonstrated "inadequate understanding of mathematical concepts and a consequent inability to apply familiar techniques in anything but the most familiar of contexts and presentations" (State Examinations Commission, 2005, p. 49). Pupils were procedurally competent but many struggled to apply procedures in novel situations and to demonstrate conceptual competence.[8] Pupils who struggle with conceptual understanding in mathematics reflect poorly on an education system that aspires, from primary level onwards, to develop pupils' abilities to understand, reason, communicate, and solve problems. Moreover, if graduates of the system who become teachers lack conceptual understanding of mathematics, they in turn will find it difficult to promote and develop conceptual understanding among the pupils they teach. Without effective intervention in this cycle it is difficult to see how mathematical understanding among pupils can be improved.

1.3 Factors that Influence Mathematics Achievement

Many variables have been considered in attempting to understand patterns of Irish pupils' mathematical achievement, including pupils' demographics, pupils' academic characteristics and behaviour, school attendance, participation in extra classes, pupils' perceptions of mathematics, family characteristics, home resources and activities, in-career development for teachers, time spent teaching mathematics, classroom resources, class size, use of technology, school size, school gender composition, school status, school location, percentage of pupils whose first language is neither Irish nor English, home-school links, provision of learning support and resource teaching, out-of-school activities, and time spent doing paid work (Cosgrove, Shiel, Sofroniou, Zastrutzki, & Shortt, 2005; Eivers et al., 2007; Surgeoner, Shiel, Close, & Millar, 2006). The influence of the different factors on students' achievement varied; more details can be found in the studies listed.

1.4 Irish Teachers' Mathematical Knowledge

Despite the range of variables that have been examined, few reports have written about the knowledge held by practising teachers. One early exception is a report on Irish teachers' mathematical knowledge from the 1920s.

[2] http://www.entemp.ie/publications/science/2006/sciencestrategy.pdf Accessed on July 10, 2008.
[3] http://www.skillsireland.ie/press/reports/pdf/egfsn080623_future_ict_skills.pdf. Accessed on July 10, 2008.
[4] http://www.education.ie/home/home.jsp?maincat=&pcategory=10861&ecategory=10876&sectionpage=12251&language=EN&link=link001&page=1&doc=39771 Accessed on July 10, 2008.
[5] http://www.education.ie/home/home.jsp?maincat=10861&pcategory=10861&ecategory=10876&sectionpage=13637&language=EN&link=link001&page=1&doc=33111, Accessed on July 10, 2008.
[6] Source: http://www.examinations.ie/archive/examiners_reports/cer_2005/LCMathematics.pdf, accessed on July 2, 2008.
[7] http://www.examinations.ie/statistics/statistics_2007/LC_2007_breakdownResults_10_or_More.pdf, accessed on July 2, 2008 and http://www.irishtimes.com/newspaper/opinion/2008/0813/1218477448360.html accessed on August 17th 2008.
[8] Partly in response to this, the National Council for Curriculum and Assessment (NCCA) announced the Project Maths initiative, which began in September 2008 which includes as an aim "the development of higher order skills, including logical reasoning and problem solving."
Source: http://www.ncca.ie/uploadedfiles/mathsreview/PMaths_En.pdf. Accessed on July 5th 2008.

At that time, a conference was summoned to report to the Minister for Education about the suitability of the National Programme of Primary Instruction. Among the group's recommendations was one declaring that "the present state of mathematical knowledge among women teachers left us no alternative but to suggest that both algebra and geometry be optional for all women teachers" (National Programme Conference, 1926, p. 12). An additional recommendation suggested that teachers' notes for mathematics should be "worded in language as un-technical as possible so that teachers, especially the older ones, may be helped and not puzzled and frightened, as many of them appear to be" by the notes that were in use at the time (pp. 16-17). Although viewing the problem as one that concerned mostly "women teachers" and "the older ones" likely oversimplified the issue, the report is one of the few reports that acknowledged the importance of teachers' mathematical knowledge.

More recent studies have investigated the mathematical knowledge held by student teachers in Mary Immaculate College, Limerick and St. Patrick's College, Dublin. The studies by Wall (2001), Corcoran (2005), Hourigan and O'Donoghue (2007) and Leavy and O'Loughlin (2006) identified shortcomings in the mathematical knowledge of several prospective teachers. The shortcomings related to specific topics such as the mean (Leavy & O'Loughlin, 2006), operations with decimals (Hourigan & O'Donoghue, 2007) and procedural and conceptual understanding (Corcoran, 2005). Corcoran (2008) further found that many student teachers were reluctant to have their mathematical knowledge audited. In each study at least one of the researchers was a teacher educator who works fulltime with prospective teachers, and their studying the topic is likely a symptom of their concern about teachers' mathematical knowledge. But such studies looked only at prospective teachers and it is possible that with some teaching experience teachers quickly gain the knowledge needed for teaching. Furthermore, the studies did not relate shortcomings in mathematical knowledge to problems in the quality of classroom instruction.

One study attempted to study the relationship between teacher knowledge and classroom instruction. The study by Greaney, Burke and McCann (1999) investigated whether Department of Education and Science (DES) inspectors considered prospective teachers who had studied mathematics as an academic subject in college to be better at teaching the subject than their peers who had studied other subjects. The researchers found that teachers who had studied mathematics to degree level were perceived to be no better at teaching the subject than teachers who had studied other subjects to degree level. The numbers who studied mathematics, however, were small (17 in one dataset and 11 in another). In addition, teachers were rated by DES inspectors on their "teaching performance relative to other teachers" (p. 27) and it is possible that criteria for rating teachers may have varied among inspectors. Such variation could have affected the findings. Nevertheless, the findings are largely in line with U.S. study results which found that beyond a certain level, university mathematics courses taken by teachers have little impact on how they teach (Borko et al., 1992) and on their pupils' mathematics achievement (Begle, 1979). These findings may be used to claim that teachers' mathematical knowledge matters little as a factor in raising pupil achievement; but it seems counter-intuitive to suggest that a teacher's mathematical knowledge is unrelated to classroom instruction and pupil achievement. Indeed, at least two recent Irish studies have suggested that primary teachers need more mathematics content knowledge (Department of Education and Science, 2002; Expert Group on Future Skills Needs, 2008). Moreover, research over the last two decades has produced new insights into the relationship between teachers' mathematical knowledge and pupil achievement that may explain previous problems in relating mathematical knowledge to the quality of instruction and pupil achievement.

1.5 International Research on Teacher Knowledge

In the mid 1980s Shulman (1986) reinvigorated research on teacher knowledge in general when he described it as the "missing paradigm" in most research on teaching. He identified three categories of content knowledge needed by teachers; the one that attracted most attention was "pedagogical content knowledge" a combination of knowing the subject and knowing "ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Such knowledge differs from the kind of content knowledge typically learned on university courses because it is knowledge that is specialised to the work of teaching. Several researchers used Shulman's ideas to study teacher knowledge in all school subjects, including mathematics. Among those researchers in mathematics were Borko (1992), Even and Tirosh (1995), Leinhardt (e.g. Leinhardt, Putnam, Stein, & Baxter, 1991; Leinhardt & Smith, 1985; Leinhardt, Zaslavsky, & Stein, 1990) and many others. Much of this research was synthesised and developed by Ball (a teacher and teacher educator), and Bass (a research mathematician), and their research colleagues at the University of Michigan. These researchers studied the work of mathematics teaching from a mathematical perspective. The teaching they studied included records of practice gathered from a year Ball spent teaching third grade pupils, where every lesson was videotaped and other records, including pupil work and teacher notes were collected.

1.6 “Mathematical Knowledge for Teaching”

Studying the practice of teaching from a mathematical perspective produced insights into the mathematical work of teaching. At the heart of the work by Ball, Bass and their colleagues is the idea of “mathematical knowledge for teaching”, a special kind of knowledge that teachers need to do the work of teaching. This knowledge differs from the knowledge that would be included in a typical university mathematics course. Ball and Bass suggest that mathematical knowledge for teaching, or MKT, consists of four domains. These domains include two types of content knowledge: common content knowledge (CCK) and specialised content knowledge (SCK); and two refinements of pedagogical content knowledge: knowledge of content and students (KCS) and knowledge of content and teaching (KCT - Ball, Thames, & Phelps, in press). CCK is knowledge that teachers hold in common with people who use mathematics in other settings; SCK is knowledge that is specialised to the work of teaching and not knowledge that people in other occupations would be expected to hold; KCS is a combination of knowing mathematics and knowing students and typical misconceptions students have; finally, KCT combines knowing mathematics and knowing teaching (Ball et al., in press). Ball and her colleagues have summarised these domains in a diagram (see Figure 1.1) with two other hypothesised domains, knowledge at the mathematical horizon and curricular knowledge.

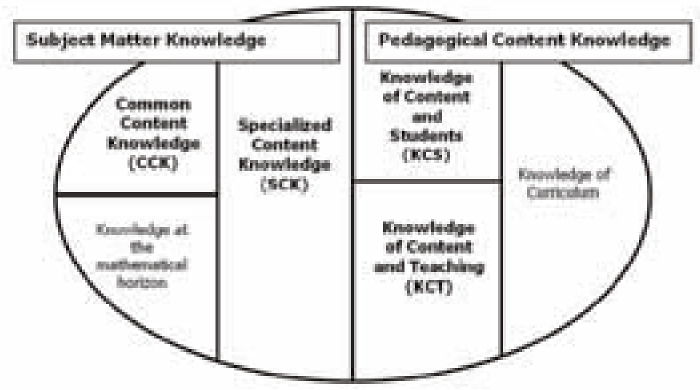
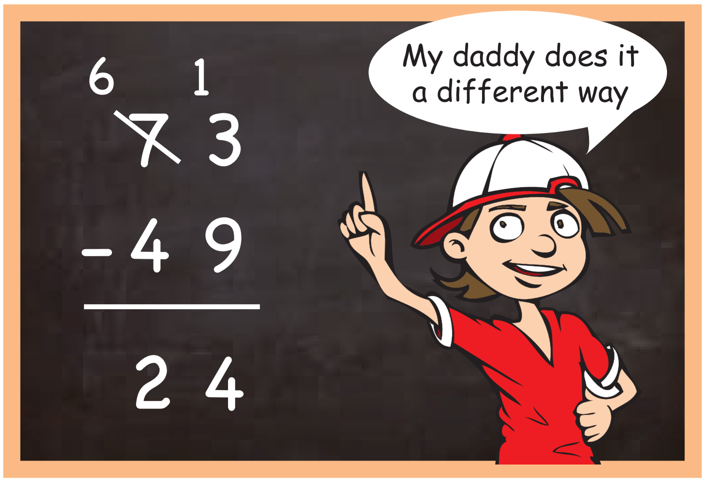


Figure 1.1.
Domains of MKT (From Ball, Thames, & Phelps, 2008). The lighter type face indicates domains of MKT that are provisional in nature.



Specific examples will illustrate how the domains of MKT support classroom practice. Imagine a teacher who is working with pupils on the topic of subtracting two-digit numbers with renaming, such as 73 – 49. A teacher uses CCK to know that the answer to this subtraction calculation is 24. Nurses, shopkeepers, accountants and other workers who use mathematics must also know how to figure out this answer. However, a teacher uses SCK to know how to respond to a pupil who says that a parent uses a different method of subtracting than the one demonstrated by the teacher. For example, the teacher may have taught subtraction using regrouping, and a parent might have demonstrated the equal additions (or “borrow or pay back”) algorithm. The teacher must understand the differences between both algorithms and how one might help or hinder pupils’ understanding of the other. This knowledge is not used by workers who use mathematics in other fields. A teacher may also draw on KCS to predict or quickly ascertain why a pupil would give the incorrect answer 36 to the problem.[9] Finally, a teacher may use KCT to decide whether the problem might best be represented for pupils using counters, base ten materials, a number line, a word problem, or a combination of these representations.

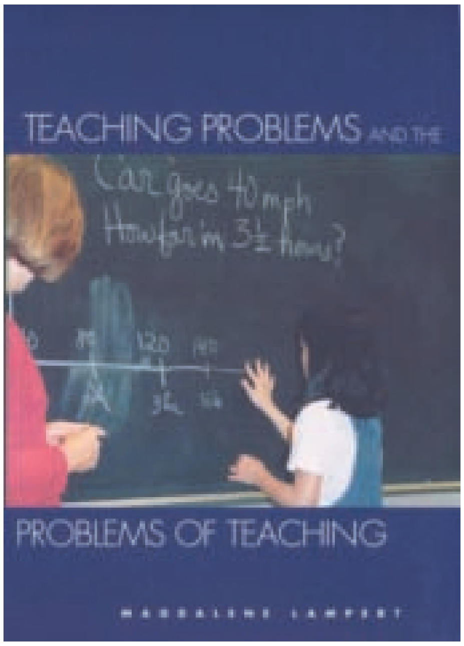
1.7 How Teacher Knowledge Affects Teaching

1.7.i High Teacher Knowledge Enhances Instruction

The studies undertaken by Ball, Bass and their colleagues, and the studies on which their work builds, have revealed many areas where teachers’ knowledge comes into play in teaching. Leinhardt has been studying teacher knowledge for many years. She and her colleagues claim that teachers with “expert” knowledge have mental plans – called “agendas” – for their lessons in which the logical sequence of a lesson is built around an overarching goal for the lesson and the connection of the lesson to previous lessons is apparent (Leinhardt et al., 1991). The same researchers claimed that the expert teachers’ lesson agendas

[9] A common error for pupils to make would be to say “3 take away 9 I cannot do, so 9 take away 3 is 6.” The pupil would then subtract the tens as if no change was necessary.

are backed up by a "curriculum script" on which teachers can draw as lessons unfold. Leinhardt has studied how teacher knowledge enhances teachers' explanations and how they represent mathematical ideas for pupils. Expert explanations were more likely than those of novices to be complete, and to contain critical features, and were less likely to contain errors (Leinhardt, 1989). Teachers' knowledge becomes apparent in choosing and using representations – analogies, pictures or manipulatives. Teachers with expert knowledge know which representations are best in teaching a particular topic but such teachers are also better judges of when a representation is no longer helpful to pupils (Leinhardt et al., 1991).



Magdalene Lampert (2001) studied her own mathematics teaching over the course of a year. Using mathematical knowledge to analyse her teaching helped her realise that much of the teaching content was unified by the concept of multiplicative relationships. Very often topics such as division and remainders, fractions and decimals, and rate and ratio are taught as if they are stand-alone topics. Lampert recognised the big concept in her teaching which helped her to connect "ideas coherently across problem contexts," elaborate ideas in new ways, and monitor "pupils' understanding and mastery of ideas and topics" (p. 261). Lampert used many other instances to illustrate how mathematical knowledge can enhance teaching, from preparing lessons to leading whole-class discussions; from teaching while pupils work independently to teaching the nature of accomplishment; and from establishing a classroom culture to teaching closure.

In another study, Swafford, Jones and Thornton (1997) provided a group of teachers with a course in geometry and a seminar on stages in pupils' learning of geometry. They subsequently investigated the effect of these interventions on a pre- and post-test of the teachers' geometry knowledge, on lesson planning, and on the teachers' instruction during the subsequent school year. Teachers who had participated in the study

a. were spending more time and more quality time on geometry instruction;
b. were more willing to try new ideas and instructional approaches;
c. were more likely to engage in risk-taking that enhanced pupil learning; and
d. were more confident in their abilities to provoke and respond to higher levels of geometrical thinking. (p. 476)

Although the researchers did not differentiate between the effects of the increased knowledge of geometry and the increased knowledge of stages in pupil cognition, it is likely that increasing the teachers' subject matter knowledge contributed to at least some of the positive outcomes on instruction reported in the study.

Studies by Ball and Bass and Hill reveal other examples of how mathematical knowledge enhances instruction. For example, Hill and her colleagues (2008) describe a teacher with high mathematical knowledge who provided constant opportunities for her pupils to share their mathematical thinking with the class; made connections between representations; explicitly described mathematical skills for her pupils; was careful in her use of mathematical language; provided a definition that was mathematically precise and comprehensible to her pupils; exhibited a commitment to teaching for equitable outcomes among her pupils; made few mathematical errors; and encouraged her pupils to use multiple solution methods.

1.7.ii Low Teacher Knowledge Constrains Instruction

The studies listed above offer examples of how teachers' mathematical knowledge can enhance their instruction. Many other studies of teaching have shown how instruction can be restricted or compromised by a teacher's lack of knowledge. Stein, Baxter and Leinhardt (1990) described a case of a teacher whose restricted mathematical knowledge resulted in his overgeneralising a limited rule and defining a function in a way that constitutes a fragile base for future learning of the topic. The same teacher missed opportunities for linkage within the mathematics topic being taught and among the representations of functions being used. In another study, Heaton (1992) described a dedicated, interested teacher who offered a mathematically inappropriate analogy for an inverse function and who reduced the mathematical content of a potentially mathematically rich and interesting class activity. Heaton attributes the problems to the teacher's not understanding the topic she was teaching. In a review of the case studied by Heaton and three other cases, Putnam and his colleagues (1992) suggested that if teachers do not fully understand the content they are teaching, they are likely to accept problem solutions that make no sense mathematically.

Schifter tells the story of a teacher whose lack of mathematical knowledge caused her difficulties when writing a word problem to match the calculation $^1/_5 + ^2/_5$. By building the problem around $^1/_5$ of the boys in her class and $^2/_5$ of the girls, the teacher varied the whole unit and the resulting sum did not relate to $^3/_5$ of the whole class as the teacher had expected (Schifter, 2001). In another example Peterson described a teacher who saw problem solving as a dispensable part of her lessons and who kept classroom discourse to a minimum. Peterson (1990) attributed these features of the teaching to the teacher's lack of knowledge about mathematics. A teacher described by Cohen (1990) taught a lesson that may have impressed a casual observer with the use of game-like activities and concrete materials. But the entire focus of the teaching was on the activities and pupils had "few opportunities…to initiate discussion, explore ideas or even ask questions" (p. 322). Cohen concluded that the teacher's "relatively superficial knowledge of [mathematics] insulated her from even a glimpse of many things she might have done to deepen pupils' understanding" (p. 322). The chasm between the teacher's frequent use of materials in teaching mathematics and the limited learning opportunities she could generate for her pupils using the materials, seem noteworthy in Ireland where an early evaluation of the mathematics curriculum by the Department of Education and Science suggests that a "broad range of mathematical resources to assist pupils' learning" now exists in most classrooms (2005a, p. 31). In the study described earlier by Hill and colleagues (2008) a teacher with low mathematical knowledge was also discussed. The teacher made frequent mathematical mistakes; her use of mathematical language lacked care and precision; and important mathematical ideas and problems were proceduralised. Several opportunities arose for pupil misunderstanding and confusion.

1.8 Mathematical Knowledge for Teaching and Student Achievement

The examples above from the body of research on teachers' mathematical knowledge suggest ways in which a teacher's having high mathematical knowledge can enrich instruction and having low mathematical knowledge can constrain instruction. Ball and her colleagues have taken this work a step further in the United States and investigated the link between teacher knowledge and pupil achievement. They did this by administering multiple choice measures[10] of mathematical knowledge for teaching to teachers and by examining the gain scores in mathematics achieved by pupils taught by those teachers over one year. They found that being taught by a teacher who scored in the top quartile of teachers as opposed to being taught by a teacher with an average MKT score, as measured by the multiple choice items, had the same effect on pupils' gain scores as if the pupils had spent an extra two to three weeks in school that year (Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005). This finding was important because it showed that mathematical knowledge for teaching made a difference in pupil achievement in mathematics.

1.9 A Context for Studying Irish Teachers' Mathematical Knowledge for Teaching

All in all the studies mentioned above provide compelling evidence of how low mathematical knowledge among teachers can constrain the quality of instruction provided, whereas high mathematical knowledge has been associated with a higher quality of mathematical instruction and with higher pupil achievement. Of course mathematical knowledge is not the only factor that matters in providing high quality mathematics instruction and in raising pupil achievement. Factors such as teacher beliefs about how mathematics should be learned, beliefs about how to make learning mathematics enjoyable for pupils, and teacher beliefs about textbooks and how they should be used have been identified as important factors (Hill et al., 2005). But given the fact that teacher knowledge has occupied a peripheral position in mathematics education research in Ireland to date, this report takes a more detailed look at the mathematical knowledge held by Irish teachers.

The mathematics standard required for entry into primary teaching in Ireland – D3 on either the ordinary or higher level paper – is relatively low.[11] Furthermore, on entry to the colleges, requirements to study mathematics vary. In Mary Immaculate College and in St. Patrick's College B.Ed. students may opt to study degree level mathematics for one or three years but not all students take this option. Although students study mathematics methods, no other subject matter study of mathematics is required. Students study mathematics methods but not mathematics content on the postgraduate programmes in Mary Immaculate College and St. Patrick's College. In the colleges associated with Trinity College Dublin – Church of Ireland College of Education, Coláiste Mhuire Marino and Froebel College – students are required to study mathematics content for two years, and a combination of mathematics methods and mathematics content for a further year. Although the mathematics courses across the three colleges associated with Trinity College Dublin follow a similar course template, the emphases vary from college to college. Postgraduate students in Coláiste Mhuire Marino and Froebel College study mathematics and mathematics methods throughout the 18-month course. In the inservice programmes delivered by the Primary Curriculum Support Programme (now the

[10]See Examples in Figures 2.1, 4.3 and 4.9.

[11]Source: http://www.education.ie/home/home.jsp?maincat=&pcategory=10900&ecategory=19312&sectionpage=12251&language=EN&link=link001&page=1&doc=16908 and http://www.education.ie/home/home.jsp?maincat=&pcategory=10900&ecategory=19312&sectionpage=12251&language=EN&

Primary Professional Development Service) to practising teachers the focus was on the new methodologies and not on developing the teachers' subject matter knowledge (Delaney, 2005).

Although obvious solutions might be to raise the mathematics entry requirement to the colleges or to extend the academic mathematics programme to all students, such measures may not deliver the desired results of improving instruction and raising pupil achievement. Research in the United States suggests that teachers need a special type of mathematical knowledge for teaching, not necessarily the kind of mathematics that is taught in secondary school or on traditional university mathematics courses. Therefore, teachers' mathematical knowledge needs to be explored at a deeper level. In the next chapter the theory and the construct of mathematical knowledge for teaching are looked at in more detail. In particular, the case will be made that primary school teaching is work that makes high demands on teachers' mathematical knowledge.

2.1 The Mathematical Work of Teaching Determines the Mathematical Knowledge for Teaching

The theory of mathematical knowledge for teaching was developed at the University of Michigan and it is based on the idea that the mathematical knowledge that teachers need is determined by the work of teaching. Ball, Bass and their research colleagues studied records of the work of teaching – teacher notes, videos of lessons, copies of pupils' work and so on – from a mathematical perspective and concluded that teaching is mathematical work (Ball & Bass, 2003) and that in order to do the work, teachers need to possess mathematical knowledge for teaching (MKT). They identified many mathematical tasks in which teachers engage, such as designing mathematically accurate explanations, representing ideas carefully, and interpreting and making judgments about pupils' questions, solutions, problems and insights (Ball & Bass, 2003, p. 11). By mathematically analysing the tasks of teaching, the researchers were able to identify mathematical knowledge needed by teachers to do the work of teaching.

2.2 Measures of Mathematical Knowledge for Teaching

Ball, Bass and Hill established the Learning Mathematics for Teaching research team – of which the author is a member – to develop measures of mathematical knowledge for teaching. These measures can be used for many purposes including: evaluating professional development for teachers, informing teacher education, and identifying tasks that are easy or difficult for particular groups of teachers. The MKT items are designed to tap into knowledge held by teachers and they are embedded in teaching contexts. An example can be seen in Figure 2.1.

The item in Figure 2.1 is set in the context of a teacher diagnosing pupils' errors. A respondent to the item is asked to adopt the role of a teacher identifying errors in pupils' work and classifying similar errors in order to target instruction to support what particular pupils need to learn. Responding to this question demands mathematical knowledge because the teacher needs to know what the correct response is in each case and to figure out what each pupil did wrong. The teacher then needs to decide which responses can be attributed to pupils making the same error. This type of mathematical analysis is specialised to the work of teaching.

Mrs. McKenna is planning mini-lessons for pupils focused on particular difficulties that they are having with adding columns of numbers. To target her instruction more effectively, she wants to work with groups of pupils who are making the same kind of error, so she looks at some recent classwork to see what they tend to do. She sees the following three pupil mistakes:

I) $\begin{array}{r} 38 \\ 49 \\ \underline{+\ 6_{1}5} \\ 142 \end{array}$ II) $\begin{array}{r} 45 \\ 37 \\ \underline{+\ 2_{1}9} \\ 101 \end{array}$ III) $\begin{array}{r} 32 \\ 14 \\ \underline{+\ 1_{1}9} \\ 64 \end{array}$

Which have the same kind of error? (Mark ONE answer.)

a I and II

b I and III

c II and III

d I, II, and III

Figure 2.1.
Sample multiple-choice item developed by the Learning Mathematics for Teaching research team at the University of Michigan and translated for use in Ireland. Original item is released and available at http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf.

Another item can be seen in Figure 2.2. This item is set in the context of a professional development workshop where teachers are asked to study four representations of the fraction multiplication sentence $1^{1}/_{2}$x $^{2}/_{3}$= 1. One of the four representations is considered to be an inappropriate representation of the problem and the teacher's task is to identify which one. Teachers must relate each pictorial representation to the numerical representation of the problem and see how each factor and the product are represented. All items were developed in the United States based on the research team's knowledge of the work of teaching mathematics in that country. But teaching in Ireland might be different because several scholars have argued that teaching is a cultural activity (e.g. Stigler & Hiebert, 1999). To address this concern the author embarked on a study of teaching observed in lessons taught by a sample of Irish teachers.

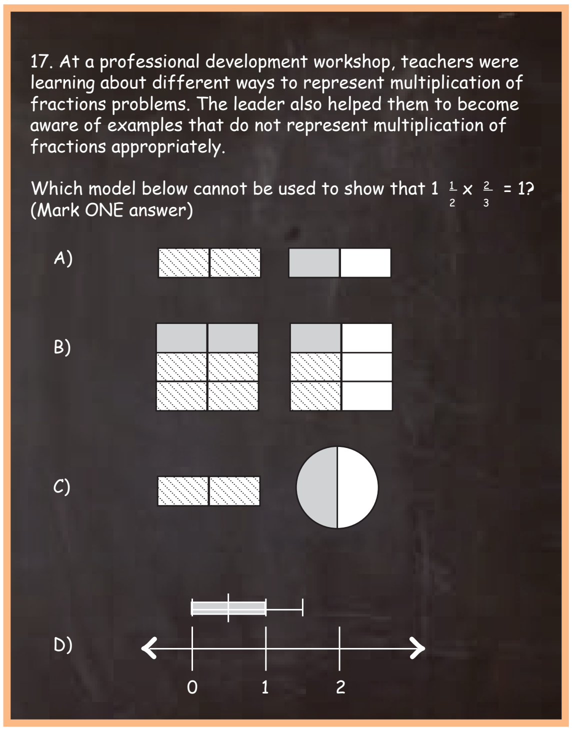


Figure 2.2.
Sample multiple-choice item developed by the Learning Mathematics for Teaching research team at the University of Michigan. Original item is released and available at http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf.

2.3 Mathematical Tasks of Teaching Observed in Irish Classrooms

Ten classrooms were visited and four mathematics lessons taught by each of the ten teachers were videotaped. Some teachers, whose names were suggested by teacher educators and by principals, were approached and other teachers volunteered to participate when they heard about the study. One lesson taught by each of the teachers was used to study the kind of mathematical work that Irish teachers do. Like Ball and Bass found in the United States, Irish teachers engaged in a substantial amount of mathematical work – work where the teacher used, or could have used, mathematical knowledge. Some examples of this work will be described in order to illustrate the mathematical knowledge that teachers use when teaching mathematics. Over 100 tasks of teaching which demand mathematical knowledge were identified. Ten of these tasks are described in some detail below and additional tasks are listed in Appendix 1. The mathematical tasks of teaching described are:

(1) Representing mathematical ideas
(2) Eliciting properties of numbers and operations
(3) Following and evaluating pupils' explanations
(4) Interpreting pupils' utterances
(5) Eliciting different ways to solve a mathematics problem
(6) Anticipating difficulties pupils will have
(7) Drawing mathematical diagrams on the board
(8) Selecting examples
(9) Connecting mathematics to the pupils' environment
(10) Deciding which pupils' ideas to take up and which to set aside.

2.3.i The Mathematical Work of Representing Mathematical Ideas

In the first example, Brendan,[12] a sixth class teacher, is teaching his pupils how to divide a whole number by a unit fraction (e.g. 7 ÷ ½ or 3 ÷ ¼). Brendan asked one pupil to draw a diagram on the board to represent the calculation 1 ÷ ¼. The pupil went to the board, drew a square and partitioned it as in Figure 2.3.

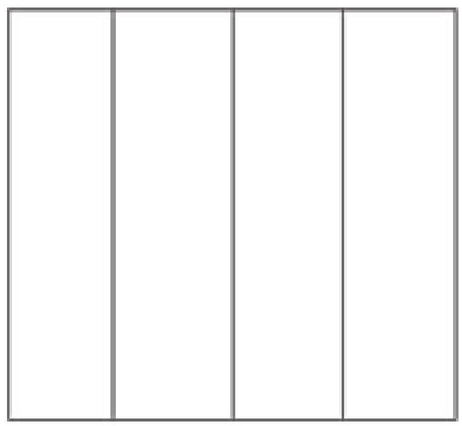


Figure 2.3.
Representation drawn by a pupil to represent 1 ÷ ¼

After drawing the square, the pupil pointed out that the square represented a whole and that you divide it into four. The pupil then hesitated and said that he didn't "see" how to draw it. Brendan asked the whole class "Is that one divided by a quarter? Is that one divided by four?" Pupils' answers were mixed so Brendan related it to division with whole numbers. He pointed out that the question is "how many quarters are in one?" and stated that "it is effectively dividing by four, isn't it?" Brendan sensed that the pupil was unhappy with the representation he had drawn and Brendan asked "are you happy with that drawing?" The pupil replied,

> Yeah, it's just the answer is all of them, not just one. It's usually one, because if you're quartering it, the answer is one of them, but if you're dividing by a quarter it's all of them, so that's what I was drawing the other way.

[12] Actual names and identifying details have been changed.

This pupil's comment illustrates the kind of mathematical knowledge that a teacher needs. The teacher must navigate between two mathematics problems that are distinct, but easily confused. One is to find a quarter of one, or 1 ÷ 4, and the other is to find how many quarters in one, or 1 ÷ ¼. This is difficult for a teacher who wants to use diagrams to represent each problem. The teacher needs to be careful that pupils do not confuse the problems. The pupil pointed out that the answer to the first calculation, 1 ÷ 4, is represented by one of the four sections of the square (¼) but for the second, the answer (4) is represented by all four quarters. In this teaching episode the teacher draws on MKT to understand a pupil's diagrammatic representation of a fraction calculation, to hear and interpret what the pupil is saying and to differentiate between two problems that seem similar but are different. If the teacher is not explicit about the differences, pupil misunderstanding may occur.

2.3.ii The Mathematical Work of Eliciting Properties of Numbers and Operations

Another example of mathematical knowledge needed for teaching occurred at the junior end of the school. Linda was teaching the number seven to her senior infants and decided to introduce the property of seven being an odd number. Linda first reviewed all the different pairs of numbers that add to make seven (7 + 0 = 7, 6 + 1 = 7 and so on). The following exchange ensued:

T: How many ways are there of making seven?

S: Eight, there's eight

T: Eight ways of making seven. Do we have a double in seven, where there are the same numbers on both sides?

S: No

T: Do we have a double? Why don't we have a double?

S: Because there's (unclear) three

T: Remember we were sharing out the teddies?

S: It's a [*sic*] odd number

T: It's an odd number. When we were sharing out the teddies we couldn't, no matter how we tried, we couldn't share them out so that the two boys had the, both had the...?

S: Same

T: Because seven is an odd number. It's not even, like number …?

S: Six

T: Six, where we had three plus …

S: Or eight.

T: Three, or eight. Exactly. It's an odd number so there are no doubles, but there are lots of pairs.

The exchange began with Linda reviewing the pairs of numbers that add to make seven and drawing pupils' attention to a type of pair which was absent – a double where both addends were the same. Linda asked the pupils why there was no double and reminded them of a previous activity where they had shared out teddies. One pupil then suggested that seven is odd. In this classroom an odd number is defined as a number where "no matter how we tried, we couldn't share them out so that the two boys had the … same." This can be expressed mathematically as 2k + 1 where there will always be one teddy left over when the set of teddies is split into two equal sized groups. When the teacher mentioned that it is different to even numbers[13] the pupils suggested examples of even numbers. The teacher here drew on her mathematical knowledge in order to know that the odd and even characteristics of numbers become important in later classes and that it is worth building the foundation for this later work in senior infants. The teacher also used a working definition of an odd number that is mathematically precise and comprehensible to 5-year-olds.

[13]For alternative definitions of even and odd numbers that might be used in school see Ball and Bass (2000a).

2.3.iii The Mathematical Work of Following and Evaluating Pupils' Explanations

Explaining is both a mathematical skill and a part of the work of teaching. As well as explaining, teachers follow, support and evaluate pupils' explanations. For this example I return to Brendan's sixth class. At the start of the lesson Brendan wrote 72 ÷ 9 on the board and asked pupils what came to mind when they looked at that expression. A pupil responded that multiplication came to mind and when Brendan asked why, the following exchange took place:

S: You have to see how many times you multiply nine and it still fits into seventy-two.

S: Yeah.

T: Ok yeah. Because multiplication and division …

S: Are the same.

T: They're the same Jack?

S: Almost. Well the basics are.

T: Ok, what do you mean by that?

S: Because really all you're doing is turning the sum around and then swapping. Ok, so you could have eight times nine equal seventy-two, but in that case you just swap the sum around and seventy-two divided by nine equals eight.

The pupil, Jack, claimed that the basics of multiplication and division are the same which is correct because division can be defined as missing factors (Parker & Baldridge, 2003). Brendan pressed the pupil for an explanation and Jack tried to describe the inverse relationship between multiplication and division. But, the idea remains unclear from the pupil's explanation which includes phrases such as "turning the sum around" and "swap the sum around." For the pupil to provide an explanation that could be more easily followed by his classmates, and by Brendan, the inverse relationship of the operations needed to be highlighted. In the subsequent exchange Brendan attempted to elicit a more complete explanation:

T: Could you add anything else? If you kept going in that plan, going off the track here a little bit but…Yes?

S: There's a word to describe it, equivalent, because like…

T: Mmm, would it be equivalent?

S: No, not really

T: I know what you're thinking, and I can understand where you're coming from, I don't think equivalent is the right word though, because when we talk about equivalence, we're actually talking …

S: It's fractions

T: Well it mightn't necessarily just be fractions, but we're talking about things that are equal, aren't we? You couldn't really say that those two things are equal. They are related certainly. They have something in common. [Teacher writes "9 x 8 = 72" on board]. It's related as well, isn't it? And what about…? [Teacher writes "72 ÷ 8 = 9" on board]. They're four tables aren't they?

Although the precise term or concept that would clarify the relationship Jack noticed between multiplication and division eluded both pupil and teacher, the exchange makes clear both the necessity for a teacher to be able to follow a pupil's explanation and the demands that doing so places on the teacher's mathematical knowledge.

2.3.iv The Mathematical Work of Interpreting Pupils' Utterances

In another classroom I observed the importance of a teacher being able to listen to pupils' utterances and to make sense of pupils' "questions, solutions, problems and insights" (Ball & Bass, 2003, p. 11). In this episode, Veronica, the teacher, was discussing with her pupils the properties of 3-D shapes, especially properties of spheres and cylinders. Notice in particular the response of the second pupil in the exchange below.

T: Why could you not stack the spheres on top of each other? What would happen? Why could you not stack spheres on top of each other? Alan?

St 1: They'll all roll down.

T: They'll all roll and they'll all fall down because they're not, you can't stack them. Excellent

St 2: If you had a little, like eh, thing, a flat thing…and there's another flat thing you could stack them like that.

T: Yeah.

The teacher asked why spheres cannot be stacked on top of spheres. Before waiting for a response she followed up with a second question and repeated the first. One pupil restated the problem that the spheres would all roll down. The teacher began to explain why but instead repeated part of the question noting that the spheres cannot be stacked. No reference was made to the curved surfaces on the spheres or to the presence of flat faces on a rectangular prism. One pupil, however, uttered a statement which used the word "flat." The pupil was hesitant in what he said (judging by the irrelevant words "little," "like" and "eh" and the repeated use of the unspecified "thing") but what he said held the seeds of explaining *why* the spheres cannot stack (because two flat surfaces are needed for stacking) and the utterance had the potential to open a discussion about which shapes *have* flat surfaces because he referred to "another flat thing." The sentence as uttered by the pupil was missing mathematical terms that even a pupil in second class could be expected to know such as "face" or "cuboid" or "shape" or "three-dimensional." Despite these shortcomings, the sentence was an attempt to respond to the teacher's question and with some work by the teacher it had the potential to elicit rich discussion in the class. The mathematical work of teaching involves recognising the potential of such tentative or unclear pupil utterances and mining them for relevant mathematics to advance pupils' mathematical understanding and thinking.

2.3.v The Mathematical Work of Eliciting Different Ways to Solve a Mathematics Problem

Many mathematics problems can be approached in different ways. Teachers need to follow multiple solution strategies when assessing pupils' understanding of concepts. In another lesson, the teacher, Clíona, was working with a group of pupils to solve the problem: "if mushrooms cost €0.62 per 100g, find the price of ¼ kg of mushrooms." One pupil suggested multiplying €0.62 by two and then finding half of €0.62. Clíona commented that "there are a number of ways, why did you choose that?" to which the pupil replied "'cause … one hundred grams is sixty two cents, so look for two hundred and fifty so you… two and a half, so you want half of that." The pupil knew that ¼ kg equals 250g and that this is the same as 200g + 50g; 200g costs twice as much as 100g which costs €0.62 and 50g costs half of €0.62. Clíona then asked if the pupils could think of another way of working it out and one pupil suggested dividing €0.62 by four and multiplying the answer by ten. This method was based on knowing that one quarter of 100g is 25g and that 25g is one tenth of 250g. The teacher elicited a third method, which involved finding the cost of a kilo of mushrooms by multiplying €0.62 by ten and dividing the answer by four. The teacher concluded that "there's three ways of doing it." For most people who use mathematics in their work, solving a problem in one way is sufficient but a teacher needs to have the mathematical knowledge to understand and evaluate different proposed solution strategies.

2.3.vi The Mathematical Work of Anticipating Difficulties Pupils Will Have

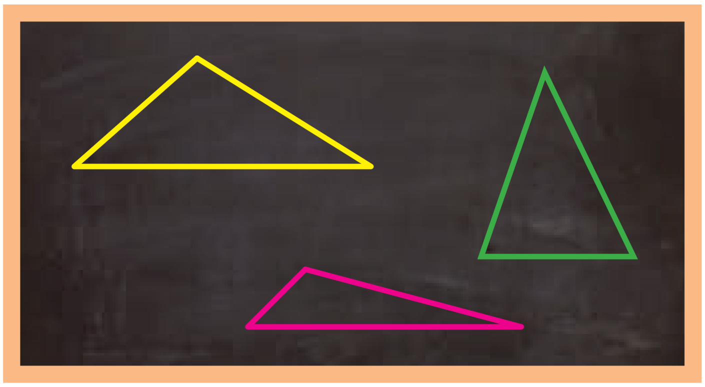
If teachers can anticipate difficulties pupils will have with particular problems, they can pre-empt those difficulties in their teaching. This was observed in a lesson taught by Eileen. The topic was to calculate how long it took a train to travel from Destination A to Destination B if it leaves A at 07:35 and arrives in B at 10:23. A common error for pupils would be to do the problem as follows:

$$\begin{array}{rcl} \cancel{1}^{9}\cancel{0} & : & {}^{11}\cancel{2}^{1}3 \\ -7 & : & 3\,5 \\ \hline 2 & : & 8\,8 \end{array}$$

In this case the pupil has over-generalised from the subtraction of numbers in the base-ten number system and has assumed that there are one hundred rather than sixty minutes in an hour. Before Eileen asked the pupils to solve this problem she cautioned them to "watch when you are doing your regrouping. Sixty minutes is not like the hundreds, tens." The teacher was drawing on her knowledge of mathematics and of pupils when she pre-empted this pupil misconception.

2.3.vii The Mathematical Work of Drawing Mathematical Diagrams

Teachers frequently need to draw diagrams on the board or on charts to illustrate various mathematical features. As well as needing suitable equipment to draw diagrams, the teacher needs to use mathematical knowledge so that the illustrations are suitable for their intended use. For example, an inaccurate circle may not make obvious the shape's symmetry; an unevenly partitioned square may not help pupils understand that both halves of a whole need to be equal in area. In one lesson, a teacher was drawing parallel lines on the board and when she was unhappy with her illustration she commented that "if I drew them straight they wouldn't" ever meet. The teacher recognised that the lines she had drawn would not provide a good illustration of the concept of parallel lines.



2.3.viii The Mathematical Work of Selecting Examples

When drawing pupils' attention to the properties of shapes teachers are encouraged to vary the types of shapes shown to pupils to help them strengthen their concepts of shapes (e.g. Clements & Sarama, 2000). One teacher made this explicit to her pupils as can be noticed in the following exchange which began with the teacher asking a pupil how many sides on an equilateral triangle:

T: But how many sides are there? Clara?

S: Three

T: Three sides. Exactly. Okay, now does a triangle have to be, do all the sides have to be equal?

S: No

T: No, because we see lots of different shapes of triangles don't we. We often see a lot of different types of triangles. Ok and if you just turn and face the white board for two seconds, I'm just going to draw up some shapes and I want you to tell me if they are triangles or not.

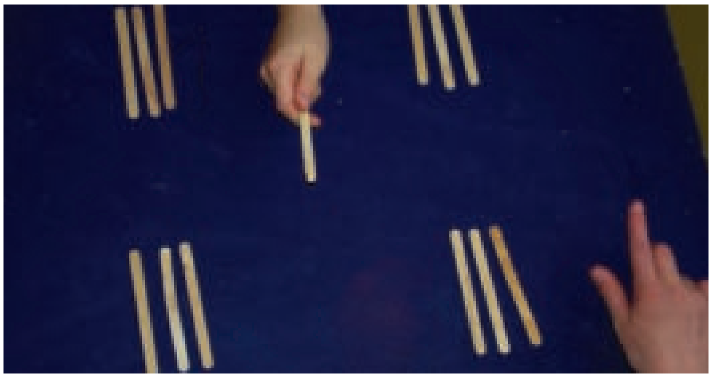
The teacher proceeded to draw various types of triangles, including scalene, on the board. Choosing such examples in mathematics class is part of the mathematical work of teaching because it requires mathematical knowledge to select shapes that can be tested by the definition of the shape but which pupils encounter less frequently than "typical examples" of the shapes.

2.3.ix The Mathematical Work of Connecting Mathematics to the Pupils' Environment

Teachers are encouraged to help students apply their mathematical knowledge in contexts related to their environment (Government of Ireland, 1999a). In one lesson pupils were converting various litre quantities into millilitres and vice versa. One pupil wrote the following "equality" in her copybook 0.25 litres = 25 millilitres. Having noticed what the pupil had written, the teacher said:

> Now we have a few problems here with this one. Nought point two five, is a quarter, isn't it? What've you written? Twenty-five. There's a huge difference between having twenty-five millilitres and two hundred and fifty millilitres. Isn't there? Two hundred and fifty is the size of that Amigo™ [teacher points to a soft drink container]. All right? Twenty-five would be, you know the, you know Calpol™. You know the little spoons you have for medicine.

In responding to the pupil the teacher attempted to relate the original quantity (0.25 litres) and the new quantity (25 millilitres) to measurement benchmarks that might be familiar to the pupil. Relating mathematics to the pupils' environment draws on the teacher's mathematical knowledge. In another lesson a teacher asked for examples of cylinders and had to decide how to respond to the suggestions of castanets and bongo drums and to the suggestion of an overhead projector as an example of a cuboid. Although these examples possess some properties of the relevant shapes, they are generally imperfect examples and the teacher's work is to reinforce pupils' learning the essential features of the shapes while relating them to accurate examples that are familiar to pupils.



2.3.x The Mathematical Work of Deciding which Pupils' Ideas to Take Up and which to Set Aside

Teachers and pupils have limited time in which to work on mathematical ideas together and in order to make the best use of their pupils' time, teachers must decide which ideas are worth pursuing and which are not. The teacher's goal is to pursue pupil comments and questions that may lead to productive work on mathematical content and skills, and to set aside ideas that may overwhelm the pupils or that may be worth deferring to another lesson. In one lesson a third-class pupil noticed that when he divided 13 lollipop sticks among four people each got three sticks and a third of one stick. But the teacher wanted to focus on the remainder of one, rather than on the fractional part so he said to the pupil: "I can see where you're coming from but don't worry, don't go there for the moment." At another stage of teaching this topic, the teacher might *want* to emphasise the relationship between the remainder of the division problem and the fraction and he might be willing to pursue the observation made by the pupil. But making these judgments requires mathematical knowledge and consequently is part of the mathematical work of teaching.

2.4 Mathematical Knowledge for Teaching: Similar in Ireland and the United States

The anecdotes given above represent a small sample of the mathematical work of teaching identified in ten Irish lessons. Other mathematical tasks of teaching were identified and they are summarised in Appendix 1. Teachers engage in additional mathematical tasks of teaching that would not be observed in videotaped lessons, such as drawing up a school plan, and reporting to parents about pupils' mathematical progress. Some examples of these tasks are included at the end of Appendix 1. The mathematical tasks of teaching identified in Ireland are broadly similar to the tasks that informed the development of the construct of MKT in the United States, which suggests that the MKT required by Irish teachers is similar to that which U.S. teachers are expected to possess.

If MKT in Ireland is similar to MKT as described in the United States, the framework of MKT is a useful one with which to study the mathematical knowledge required by Irish teachers. One instrument based on the construct is the set of multiple choice measures of MKT. Because the measures were designed for use in the United States, the items needed to be adapted for use in Ireland and this has been documented elsewhere (Delaney, Ball, Hill, Schilling, & Zopf, 2008). Some might question the use of multiple-choice questions to study a phenomenon as complex as teacher knowledge. The focus of this report was not to consider teacher knowledge as an end in itself but to describe knowledge that would make a difference in mathematics instruction. Therefore, it was necessary to validate the use of the measures for making claims about teachers' knowledge that could have an impact on instruction. The topic of validity is discussed in Chapter 3.

3.1 The "Mathematical Quality of Instruction"

Irish teachers' scores on the multiple-choice measures of mathematical knowledge for teaching would be of little interest unless the scores were related to the "mathematical quality of instruction" (Ball & Bass, 2000b) observed in lessons. The mathematical quality of instruction refers to characteristics of instruction, such as how teachers represent mathematical ideas and connect representations to each other; how they describe, explain and justify mathematical ideas and encourage their pupils to do the same; how accurately teachers use language and how explicit they are in talking about mathematical practices. In short, it refers to "several dimensions that characterise the rigor and richness of the mathematics of the lesson" (Ball & Bass, 2000b, p. 4). These characteristics are likely to be present in lessons taught by teachers with MKT and missing from lessons taught by teachers who lack MKT. The relationship between teachers' scores on the measures and the mathematical quality of their instruction was studied. Of interest was whether teachers' scores on the multiple-choice items were associated with instruction that is mathematically rich and free from errors. If such a relationship existed, the multiple-choice measures would be useful for predicting the mathematical quality of instruction among Irish teachers.

3.2 The Mathematical Quality of Instruction Observed in 40 Irish Lessons

3.2.i The Teachers

To study the relationship between scores on the MKT measures and the mathematical quality of instruction, ten teachers – eight female and two male – were videotaped teaching four lessons each. The classes taught ranged from senior infants to sixth class, and the teachers had been teaching for between 3 and 30 years. Although the sample of teachers was not randomly chosen, several school types were represented: inner city, rural, suburban, single-stream and multi-grade. Teachers in co-educational, all-boys and all-girls schools were included and some teachers taught in schools designated as disadvantaged. One teacher taught in a private school that followed the DES curriculum; the teacher was a fully recognised and probated primary school teacher who had attended professional development for the 1999 curriculum. None of the ten teachers was teaching in a gaelscoil at the time of the study.[14]

3.2.ii The Lessons

Each teacher taught four lessons. Lessons were generally taught over a two to three week period, with times agreed to suit both the teacher and the researcher. Teachers chose the topics they wanted to teach, although they were asked to include, if possible, two different topics over the four lessons. All but one teacher did this. Teachers were asked to teach lessons of a similar duration to their regular mathematics lessons. One camera was used to record the lessons and it was generally focused on the teacher.[15]

3.2.iii The Instrument Used to Code the Mathematical Quality of Instruction

When the 40 lessons had been videotaped, the quality of mathematical instruction in each lesson was analysed. The instrument used to do this was one devised by members of the Learning Mathematics for Teaching research group at the University of Michigan. The instrument consists of 32 features of mathematics instruction known as "codes" grouped in three sections and an accompanying glossary to explain the codes (Learning Mathematics for Teaching, 2006).[16] The first set of codes considers how the teacher's knowledge of the mathematics of the lesson topic is evident in the lesson. Sample codes in this set include the teacher's use of technical language, the presence of explanations, and the teacher's selection of representations and how they are linked to each other. The second category of codes refers to how the teacher uses mathematics with pupils. Sample codes include how the teacher responds to pupils' errors, how mathematical work is recorded in class and whether the teacher elicits explanations from pupils. The third set of codes considers how the teacher uses mathematical knowledge to teach equitably, so that pupils of all races and social classes are included and can participate in the lesson. Codes in this category include the amount of time spent on mathematics, the teacher's encouragement of a diverse array of mathematical competence and the teacher's explicitness about language and mathematical practices. Finally, coders rated the teacher's knowledge as low, medium or high based on the entire lesson.

3.2.iv The Procedure for Coding Lessons

For coding purposes lessons were divided into five-minute clips. Two experienced coders, primarily members of the Learning Mathematics for Teaching research team, were randomly assigned to code each lesson.[17] Each coder watched the lesson through and then watched the lesson again, stopping to code each five-minute clip. Both coders subsequently met to reconcile codes and they supplied an agreed set of codes for each lesson, which became the record of the mathematical quality of instruction in the lesson.

When coding each lesson clip, a number of decisions had to be made. The decision process will now be described with reference to one code: a teacher's use

[14] The reason for this was that some U.S. researchers were assisting with analysing the data and therefore all lessons needed to be taught through the medium of English.
[15] Written permission was sought from the teacher, the school principal and parents of the pupils in the classroom. Pupils whose parents did not give permission for them to be recorded sat outside the range of the camera.
[16] For more information see http://sitemaker.umich.edu/lmt/faq_about_video_codes
[17] This procedure was followed for 70% of the lessons and the author, who is a member of the Learning Mathematics for Teaching team, coded the remainder of them alone.

of conventional notation or mathematical symbols. A coder first decided whether a feature, in this case the use of conventional notation, was "present" or "not present" in a lesson clip. If the teacher wrote the numeral "4" or the word "parallelogram" on the board, a coder may wonder whether they count as mathematical symbols. The glossary clarifies that "by 'conventional notation,' we do <u>not</u> mean use of numerals or mathematical terms"[18] so if no other notation appeared, the relevant category code for the clip would be "not present." The second decision to be made was whether the presence or absence of a feature was appropriate or inappropriate. If, for example, conventional notation was present and mathematically accurate, it was marked as "present and appropriate." On the other hand if a teacher recorded on the board a statement such as the following: 7 + 6 = 13 + 5 = 18, it was coded as "present" because it includes the "addition" and "equals" mathematical symbols. But the statement is inaccurate because $7 + 6 \neq 13 + 5$ so it would have been coded as "inappropriate." The overall decision in this case, therefore, would be "present – inappropriate." If the absence of an element seems appropriate, it is coded "not present – appropriate" or if the absence seems problematic it is coded as "not present – inappropriate." A typical cell to be completed for each code is represented in Figure 3.1.

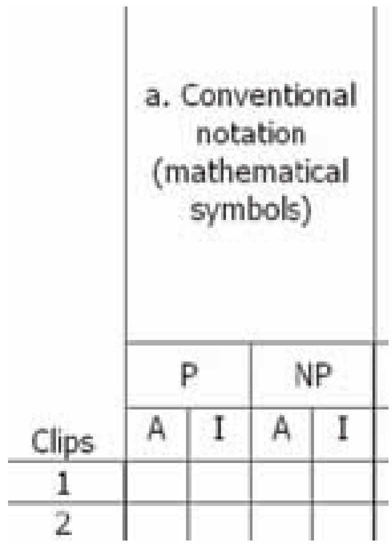


Figure 3.1.
A section of the grid used for video-coding.
(P = Present;
NP = Not Present;
A = Appropriate; and
I = Inappropriate).

3.3 Relating Teachers MKT Scores to the Mathematical Quality of Instruction

3.3.i Item Response Theory Scores

All ten teachers completed a set of multiple-choice measures of MKT under test-like conditions. When the 40 lessons had been coded and all 10 teachers had completed the multiple-choice measures of MKT, it was possible to study the relationship between the mathematical quality of instruction and teachers' scores on the multiple-choice items. A score for each teacher's performance on the multiple-choice items was calculated using item response theory (IRT). Raw scores or percentage scores are not used because such scores take no account of the relative difficulty of items. For example, two teachers may have the same percentage score but one teacher may have shown greater proficiency by answering more questions that teachers generally found to be difficult. Such differences in proficiency are concealed in raw or percentage scores. Furthermore, the MKT items are not criterion referenced so there is no expected performance level by which to judge teachers' scores so that a raw score would have little meaning. The IRT score takes into account differences in item difficulty (Bock, Thissen, & Zimowski, 1997). In this study the scale used to present teachers' scores on the MKT measures has an average of 0 and a standard deviation of 1. Thus a teacher with a score of +3 (plus three) can be considered to possess high MKT and a teacher with a score of -3 (minus three) possesses low MKT.

3.3.ii Video Teachers' Scores on MKT Measures

Although it was hoped to recruit teachers with a wide range of scores to participate in the video study, all teachers recruited were in the top two thirds of Irish teachers in terms of their scores on the MKT measures. The scores ranged from the 36th to the 97th percentile of teachers' scores as can be seen in Table 3.1. Percentiles were calculated based on the scores of the 501 teachers who participated in the national survey of MKT, which will be discussed in Chapter 4.

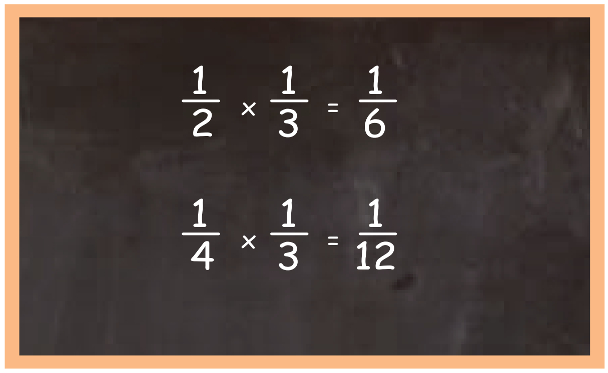
Table 3.1
Irish teachers from the video study with their MKT scores (range from -3 to +3) and their percentile in the population calculated based on all 501 teachers who participated in the national MKT study

Teacher	MKT Score	Percentile
Olive	1.88	97
Nigel	1.30	91
Brendan	1.28	90
Eileen	0.78	83
Clíona	0.68	82
Sheila	0.53	78
Veronica	0.36	57
Hilda	-0.14	46
Caroline	-0.36	42
Linda	-0.43	36

[18] The Video Coding Glossary is available at http://sitemaker.umich.edu/lmt/files/lmt-mqi_glossary_1.pdf. Downloaded on August 7th 2008.

3.3.iii Teachers' MKT Scores and Overall Mathematical Quality of Instruction

The first step in studying the relationship between teachers' scores on the MKT measures and the mathematical quality of instruction was to consider the relationship between teachers' overall scores for mathematical quality of instruction and their scores on the multiple-choice items. As stated above, each lesson was assigned an overall rating of low, medium or high based on the mathematical quality of instruction observed in the lesson. Intermediate values of low-medium and medium-high were possible. If the teachers are ordered according to their scores on the MKT measures and according to the overall mathematical quality of instruction (see Figures 3.2a and 3.2b), the teachers who scored higher on the multiple-choice items were also generally considered to exhibit higher mathematical quality in their instruction. Only one of the teachers with the top five scores on MKT – Eileen – was not in the top five in terms of mathematical quality of instruction. Among the five low scoring teachers, only Linda's instruction demonstrated higher mathematical quality of instruction than was predicted by her MKT score. Within the top and bottom bands there were some discrepancies. Clíona, for example, was considered to exhibit the highest mathematical quality of instruction but her MKT score was only fifth highest (though her percentile score was in the top quintile of all teachers) and Veronica whose mathematical quality of instruction was considered to be lowest, scored seventh overall on the measures. In order to understand this more fully, it is worth looking inside some of the classrooms.



3.4 MKT Scores Consistent with Mathematical Quality of Instruction

3.4.i Brendan – High MKT Score and High Mathematical Quality of Instruction

Both Brendan and Hilda exhibited instruction consistent with their MKT scores. Brendan's MKT score is in the 90th percentile of Irish teachers and his instruction exhibited many elements of mathematical quality. An episode

Figure 3.2a
Teachers in the video study ordered according to their IRT scores on the MKT survey (scored from -3 to +3; teachers not placed to precise scale).

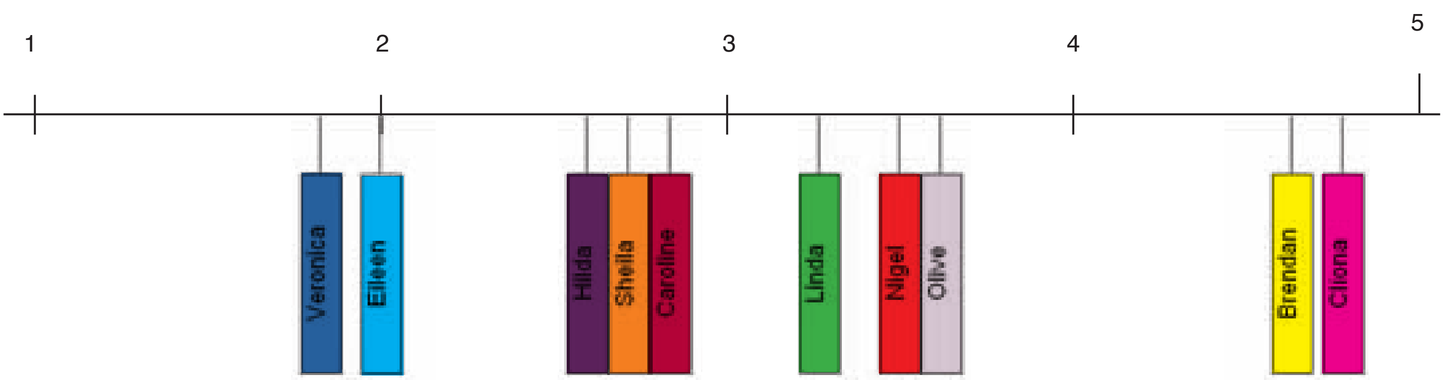


Figure 3.2b
Teachers in the video study ordered according to the overall MKT observed in their instruction, relative to other teachers in the video study (scored from 1 to 5; teachers not placed to precise scale).

from one lesson illustrates this. Brendan and his pupils were folding paper into halves, thirds or quarters and then folding them again in order to figure out answers to problems such as $^1/_2$ of $^1/_3$ and $^1/_4$ of $^1/_3$. Aided by Brendan's prompting, the pupils noticed the pattern whereby the product could be found by multiplying both fractions. The discovery was confusing for some pupils because in the paper folding activity they had been *dividing* paper but now they could solve the problems using multiplication. One pupil grappled with the apparent contradiction and asked a question:

S: Yeah, but it's also division, right?

T: Yeah, it is. Well you are dividing. What you've been doing on the page has been dividing.

Brendan agreed with the pupil that division is involved in the operation as well. This is correct because in the case of $^1/_2$ of $^1/_3$, $^1/_2$ is an operator that "stretches" $^1/_3$ one time (i.e. the size of the numerator) and "shrinks" it by dividing it by 2, the size of the denominator (Behr, Lesh, Post, & Silver, 1983). Brendan related his response to the paper folding activity to explain the division component of the calculation.

A moment later Brendan's knowledge was tested again when he asked a pupil to compute $^1/_4$ of $^1/_3$. Based on the previous exchange, the pupil asked if he would do it "as a division or multiplication sum." The following discussion took place as Brendan probed the pupil:

T: Well, is it going to work? How would you write it as a division sum?

S: You get a third and divide it by a quarter. You get a twelfth [pupil writes $^1/_3 \div {}^1/_4 = {}^1/_{12}$ on the board], so it's the same thing.

The pupil incorrectly replaced the "of" term with a division symbol and reversed the order of the fractions but he wrote the correct answer, which had been figured out previously using the paper folding activity. Based on this solution, the pupil claimed that division is the same as multiplying. Brendan knew, however, that the method used by the pupil to compute the answer was incorrect and he asked "is it though?" The pupil responded as follows:

> Because it's fractions part of it....Dividing means it gets bigger. When you divide a third by a half it gets bigger, the number. Because if it was over, if it was over one it would be, the number would get smaller…. But if it's under one it gets bigger.

The pupil's statement made further demands on Brendan's MKT because the statement required deciphering (and meanwhile other pupils were trying to contribute to the discussion). To decipher the statement Brendan needed to know that when the pupil referred to dividing making a number bigger, he is referring to dividing fractions (If you divide 3 by ½ , for example, you get 6). When the pupil referred to the number being "over one" he is referring to division of counting numbers by counting numbers. Brendan also needed to recognise that the specific fraction computation mentioned by the pupil (dividing a third by a half) was not the question the pupil was asked to work on but an example chosen by the pupil to illustrate his point. With little or no time to think, Brendan responded as follows:

> You're dead right. Maybe the way you've written it isn't exactly accurate. Do you see the third divided by a quarter? Are you dividing it by a quarter or are you dividing it by four?

Brendan's response signalled that he agreed with the pupil's explanation about dividing but the teacher also drew attention to the pupil's error by giving a clue to what was wrong: the pupil had written that he was dividing a third by a quarter but it should have been a third divided by four (because the problem required the pupil to find one-quarter of one-third). The pupil's reply revealed another misconception as evident in the subsequent exchange:

S: Same thing basically.

T: I don't think so. You're dividing into quarters, but are you dividing by a quarter?

S: Oh yeah.

The pupil had thought that dividing by a quarter was the same as finding a quarter but Brendan used his MKT to distinguish between "dividing into quarters" (i.e. dividing by four) and "dividing by a quarter." The pupil's response of "oh yeah" indicated that he realised his error. Subtle mathematical differences exist between dividing into quarters and dividing by a quarter but teachers need such knowledge. Brendan clarified what needed to be done and posed another question:

> When you're splitting something into four, you're dividing by four, aren't you? You're dividing into four pieces. That's the only thing I'd change in that maths sentence. A third divided by four. How would you write four as a fraction?

One pupil's response to Brendan's question made further demands on his knowledge: The pupil responded that four could be written as "sixteen over four" before Brendan elicited another answer, "four over one." Brendan asked why that was correct. One pupil offered an explanation, which was correct but fell short of an explanation and was difficult to follow:

> Because when you're emm, say if you're multiplying emm four by five but you want to do it in fraction term (sic), you can't emm you can't just put like, say you put five over four you can't do that, so you have to put one over it. So then it would be one eh, over four times one over five or emm… Four over one times five over one...so it'd make it easier

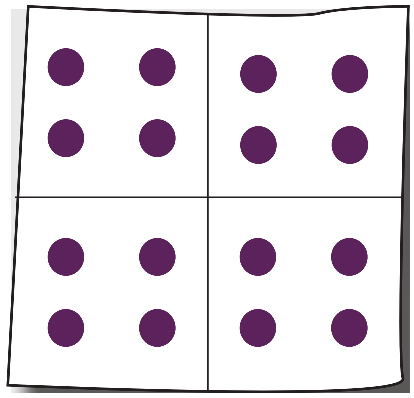
The pupil took a specific case of multiplying in fraction terms to illustrate how to write whole numbers as fractions. Brendan acknowledged being confused by the response and instead offered his own explanation:

> Well, one over one is one whole, isn't it? So, I mean, four over one is four whole amounts.

In the episode described above Brendan exhibited knowledge of fractions as operators where the operations of division and multiplication are closely related; he evaluated and responded to a pupil's incorrect answer; he deciphered a pupil's inchoate contribution; he distinguished between a pupil's oral description of a procedure and what the pupil wrote; he identified pupil misconceptions and he explained an idea. All these incidents occurred in a period of less than three minutes of a one hour lesson, showing how little time Brendan had to think about his answers. Throughout the four lessons observed, he exhibited similar knowledge making few mathematical errors and using mathematical language appropriately throughout. Both MKT and the mathematical quality of instruction were consistently high.

3.4.ii Hilda – Moderate MKT Score and Medium Mathematical Quality of Instruction

Like Brendan, Hilda's MKT score was consistent with her mathematical quality of instruction but her scores were lower than his. Hilda's MKT score was in the 46th percentile and her instruction exhibited traits of both high and low mathematical knowledge. Her use of explanations was characteristic of high MKT and she frequently asked her 2nd class pupils to explain their work. In one example pupils had folded a page into quarters and found a quarter of 16 counters by placing an equal number of counters on each quarter of the page. Hilda asked the pupils what half of sixteen would be and when a pupil answered eight, Hilda pursued the following explanation:



T: And how did you get that from what you've done here?

S: Because I had four here and I had four here.

T: Yeah?

S: And four and four equals eight.

T: Makes eight. And so what is this part of your page?

S: Half.

T: Good boy, ok. And what did we say about halves and quarters?

S: Halves are bigger than quarters.

T: They are, yeah. And two quarters is the same as a half. Yeah, well done.

In this exchange Hilda wanted the pupils to see that two quarters equal one half and together with a pupil she built an explanation of why knowing a quarter of sixteen made it possible to figure out half of sixteen. In addition, Hilda used mathematical terms appropriately in her lessons, including parallel, horizontal and symmetrical. Occasionally pupils challenged Hilda's knowledge, as they had done to Brendan, such as when a pupil claimed that a globe was an example of a circle. Hilda corrected the misconception.

On other occasions her instruction exhibited lower mathematical quality such as when she accepted a pupil's characterisation of a rectangle as having "two small sides and two long sides." This definition excludes a square, a special case of a rectangle where all sides are equal in length. In another lesson about a rectangle the following exchange occurred:

T: How many faces would it have? Ailbhe?

S: Two

T: Two faces, front and the back. So because it has two faces, what type of a shape is it? Who can tell me what type of a shape is it? Daniel?

S: 2-D.

T: Good boy, 2-D. And what does 2-D mean? 2-D shape, Joan?

S: It means that it's flat.

T: It's flat. Exactly. A 2D shape is?

S: Flat.

T: Flat exactly; because it only has two dimensions, it only has two faces, the front and the back. Whereas the 3D shape is?

S: A cube.

T: Bigger like a cube, very good, a cube or a cuboid, because it's got more faces. So that is quite important that we know the difference between 2-D and 3-D shapes, so today we're learning all about?

S: 2-D

In this interaction Hilda asked a pupil how many faces on a rectangle and Hilda agreed with the pupil's incorrect response of two. She named the faces as the front and the back of the rectangle. The error is compounded when three-dimensional shapes were contrasted with two-dimensional shapes as having more faces, rather than because they are solid shapes. This lack of knowledge resulted in Hilda conveying inaccurate information about the dimensions of shapes to her pupils. Earlier in the same lesson Hilda defined parallel as follows:

> What parallel means is that two lines are running beside each other but they will never meet. Can you see the way these two lines run straight up? Ok. They go straight and they are never going to meet because they will keep going straight. Ok. The same with these two sides, see, they are going straight beside each other but they'll never meet.

Although Hilda supplements the definition by pointing to the relevant sides of the rectangle, the definition contains terms that could be confusing for second class pupils such as "running beside each other" and "never going to meet." This is an example of a definition that might be suitable for older pupils but where some expressions render it unhelpful for younger pupils. In summary, Hilda's responses to pupils' errors had some evidence of low MKT, whereas she exhibited rich mathematics in her explanations and use of multiple representations, indicators of high MKT. Overall the mathematical quality of her instruction was consistent with her scores on the MKT measures.

3.5 MKT Scores Inconsistent with Mathematical Quality of Instruction

3.5.i Eileen – High MKT Score and Low-Medium Mathematical Quality of Instruction

In contrast, the mathematical quality of Eileen's instruction fell short of what would be expected based on her MKT score. Eileen's lower than expected mathematical quality of instruction rating may be illustrated with reference to a specific lesson. The lesson centred on a cookery theme, in which she was organising ingredients needed for a subsequent lesson. At the outset of the lesson Eileen asked the pupils how cooking "ties in with maths." Eileen agreed with several suggestions offered by pupils: weight, measurement, time, and length but challenged no pupil to elaborate on how the topics were connected to the cooking theme. She did, however, add ratio to the list but it was explained in an unclear way.

T: Ratio, how does ratio come into it?

St: Five spoons.

St: Five spoonfuls to a cupful of (unclear)

St: It's like fractions and stuff like that.

St: A teaspoonful

T: Exactly.

Eileen seemed to assume that the pupils understood potentially complicated ideas, such as ratio, and as a result she was frequently not explicit when explaining terms. Although the seed of the idea of ratio (comparison of quantities) is contained in the exchange above, for a pupil who had forgotten what ratio is or who had not understood it in the first place this exchange would hardly help. Eileen's own strong mathematical knowledge may have caused her to attribute to pupils more understanding than was justified by the evidence. She frequently accepted from pupils and offered to pupils incomplete explanations.

Using a practical approach (such as cooking) when teaching mathematics is consistent with the *Primary School Curriculum: Mathematics Teacher Guidelines* which state that "all number work should be based as much as possible on the children's own experiences and real-life examples used" (Government of Ireland, 1999b, p. 9). The limitations of using real-life examples were evident in this lesson in which pupils were distracted by the context and spent more time engaged in transcribing recipes and deciding who would bring in particular ingredients than working on mathematical skills and content. No doubt, cooking offers multiple opportunities to apply mathematics: doubling or halving quantities of ingredients, estimating and weighing, comparing prices of ingredients and so on. One practical example in Eileen's class had great potential for discussing mathematics. A recipe for a custard tart required using 250 ml of egg custard and Eileen wanted the pupils to make triple the quantity of custard. Pupils had to figure out the new quantity to be made and the necessary ingredients, based on knowing the ingredients needed to make 1000ml of egg custard. This offered a practical context in which to apply the unitary method (and other methods) but it was lost in the overall excitement of the lesson. There were other examples where Eileen attempted to be ambitious in her teaching (such as calculating probabilities when two dice were thrown) by using interesting contexts but where the mathematics the pupils were working on was obscured. Eileen chose interesting activities for her pupils and she regularly encouraged them to look up mathematical ideas in mathematics books. Problems arose when the lesson context overpowered its mathematical content and when Eileen left mathematical ideas vague or incomplete.

3.5.ii Veronica – Moderate MKT Score and Low Mathematical Quality of Instruction

Veronica was another teacher whose mathematical quality of instruction was lower than expected based on her MKT score. Several reasons may explain this. First, neither Veronica nor her pupils used a textbook in the observed lessons and this may have deprived the class of a working definition for the shapes being discussed. If accurate, comprehensible definitions of shapes had been available, Veronica may have been less accepting of some objects in the environment offered as examples of cones, cuboids and cylinders.

In addition, much time in Veronica's lessons was spent making 3-D shapes, which added little to the mathematics being taught. Such an activity is consistent with the mathematics curriculum which recommends that pupils construct three-dimensional shapes (Government of Ireland, 1999a). Observing shape construction in practice, however, prompts the paraphrasing of a question asked by Baroody (1989): Can pupils use the activity "in such a way that it connects with their existing knowledge and, hence, is meaningful to *them?* Is the [activity] used in such a way that it requires reflection or thought on the part of pupils?" (p. 4, italics in original). Evidence from the video lessons suggests that in Veronica's case the answers to both questions were frequently no, and the activities reduced rather than enhanced the mathematical quality of her instruction.

Another possible explanation for the inconsistency between Veronica's MKT score and the mathematical quality of instruction is her teaching style. She regularly encouraged pupils to contribute to classroom discussions and she enthusiastically affirmed *every* contribution. The problem was that in her enthusiasm she sometimes accepted incorrect, inaccurate or incomplete responses and seemed unwilling to challenge pupils to refine or correct what they said. Furthermore, potentially worthwhile contributions from pupils were lost in the enthusiastic and lively, but unfocused classroom discussions. In short, Veronica's lessons showed a lower quality of mathematics than expected, possibly because of one of the following factors: the lack of support that the use of a textbook would have provided; her use of activities with little mathematical merit; or her lively discussions combined with an apparent reluctance to challenge the pupils' responses.

3.5.iii Clíona – High MKT Score and High Mathematical Quality of Instruction

In contrast, the mathematical quality of Clíona's instruction, relative to other teachers in the study, was rated as somewhat higher than would be expected based on her MKT score. She had the highest overall lesson score and although her level of MKT is high compared to Irish teachers generally, it was in the middle of the ten teachers discussed here. Clíona's teaching provided opportunities for all pupils to participate in problem solving and she encouraged them to reason mathematically and to justify their responses. Clíona was careful about her use of language. She conveyed the message to pupils that they could all do the work required and that effort invested was worthwhile. An extract from one of Clíona's lessons helps explain her style of teaching. In this excerpt she referred to an activity from a previous lesson where the pupils had used string to measure the circumference of a circle and had made inferences based on the results about the relationship of the circumference to the diameter. Clíona began with a question:

T:	What did you learn from that?
S	That the diameter, that the circumference is three times bigger than the diameter
T	Very good, or approximately. It's not an exact science there. It's approximately three times greater than the diameter.
T	So Damien on that information, if I gave you the circumference of a circle how would you establish the diameter or the approximate, the approximate diameter?
S	Eh, the
T	If you have your circumference and I'm asking you to give me the approximate diameter how would you do it?
S	Eh fold that in three
T	And?
S	Eh
T	What would you have to do then Damien? You might need another bit of equipment. Can anyone help Damien?
S	Measure it.
T	Yeah, good man. Of course you'd get out your ruler and you'd measure it wouldn't you? So you're folding it in three but come on, what else could you do? What would be even easier, as a sum to do that …
S	Divide it by three
T	Good man Robert. Write down your circumference and divide it by three. And what would that give you Robert?
S	Approximately three point seven
T	No, the approximate …
S	The approximate diameter.
T	Good and how would you establish the radius then from that eh Charlotte
	What's the relationship there between the radius and the diameter?
S	Emm, you ...
T	Radius, diameter, what's the relationship?
S	Divide by two.
T	Thanks Laura. You're very good.

In this piece of classroom interaction Clíona moved from recalling a previous lesson activity, to posing questions about how to find the length of the radius of a circle. In the course of the discussion she reminded pupils that describing the relationship of a diameter to the circumference as being a third is approximate. She elicited the operation that could be used to find the length of the diameter if the circumference is known, and she established that the pupils knew the relationship of the radius to the diameter. She built on pupils' answers encouraging them to make a link between "folding it in three" and dividing by three. A few hypotheses may help explain why the mathematical quality of Clíona's instruction is higher than suggested by her MKT score. She prepared well for her lessons and frequently referred to her notes and to the textbook. In one case she says "Now children …just give me a moment now. I have it written down here somewhere, what we're going to explore," indicating that she has planned the lesson material in advance. In another lesson she referred to her notes or to a textbook when explaining the word "vertex." That explanation gives another clue as to her performance when Clíona asked the pupils for another word for corners:

T: What other word have we?

Ss: Vert….vertex…vert-ice

T: We'll get it right. Vertices. Plural. Vertices It's a Latin word. Comes from the word "vertex," is a Latin word. So it's one vertex and it's many vertices. So we've faces, we've vertices, and we have?

Clíona responds not just by telling the pupils the word but by telling them something about the word's Latin origin. Frequently in lessons she looked for synonyms (e.g. for net, and for minus five). Her interest in language generally may help to explain why Clíona was careful and precise in her use of mathematical terms and in her general language when talking about mathematical ideas. A third possible reason is her teaching situation. The class has three grades and fewer than 20 pupils in total and Clíona's interaction with the pupils was like interacting with a large family. Notice in the quotation above Clíona said "*We*'ll get it right." The impression given is of a teacher and pupils working together to learn. She asked pupils to describe steps of procedures, to explain and clarify what they meant and she responded to pupil errors by taking on board the errors and perhaps reframing the question or calling on another pupil to respond. Sometimes she made mathematical mistakes such as saying that a circle has width and not height, or she confused the mathematical meaning of edge (where two faces meet) with the everyday meaning (edge of a plate). These errors, however, appeared minor compared to the explicitness of her teaching and her encouragement of pupil effort. Factors such as detailed lesson preparation, attention to precise use of language generally and ways of probing and refining pupils' answers are unlikely to be measured by the MKT measures but in Clíona's case they enhanced the mathematical quality of instruction.

3.6 Correlation of MKT Scores and Mathematical Quality of Instruction Ratings

Despite the discrepant cases, the MKT measures were relatively effective at predicting the mathematical quality of instruction. The overall correlation between scores on the MKT measures and the ratings of mathematical quality of instruction was moderate at 0.43. Although a higher correlation would have been welcome, it is relatively easy to hypothesise why it was moderate. Six of the ten teachers were in the top quartile of Irish teachers based on their MKT score and no teacher was in the lower tercile of teachers. When teachers are located so close together on the scale, test items would need to be particularly sensitive in order to discriminate well among the teachers. One way to think about this is that a classroom mathematics balance would be a good instrument for comparing the weights of different bundles of feathers but would be less efficient at distinguishing among the weights of individual feathers. The lack of sensitivity of the MKT measures is not a problem when measuring the MKT of a large number of teachers but can be problematic when a small number is involved. Therefore, MKT scores and ratings of the mathematical quality of instruction may be inconsistent because of measurement error. Repeating the analysis of the relationship between MKT scores and mathematical quality of instruction ratings, with randomly selected teachers would be worth considering in the hope of raising the correlation between them.

In summary therefore, teachers' scores on the MKT measures are related to the mathematical quality of instruction. The relationship holds for groups of teachers – for example, in the group of ten teachers, the MKT measures predicted the half in which eight of the teachers would be placed based on the mathematical quality of their instruction. But it cannot be claimed that the relationship between MKT and the mathematical quality of instruction holds on an individual basis because discrepant cases were identified. For the purposes of this study, the MKT measures can be used to make inferences about the quality of Irish teachers' mathematics instruction generally, but in any specific teacher's case the inference may not hold.

In Chapter 2 it was shown that the construct of MKT is similar in both Ireland and the United States. Chapter 3 demonstrated that teachers' MKT results are valid for use at a large group level in that teachers' scores on the items are generally predictive of the mathematical quality of their instruction. Results of Irish teachers' performances on the items will be presented in Chapter 4.

4.1 Surveying Irish Teachers' Mathematical Knowledge for Teaching

4.1.i Composition of Items on Survey Form

This chapter presents results of teachers' performances at a national level on the multiple-choice measures of teachers' mathematical knowledge for teaching. The survey form included items on number, algebra and geometry. The items used were selected and adapted from a bank of items created in the United States (see Delaney et al, 2008), but no items related to the measures and data strands of the curriculum had been developed at the time the survey was administered. Items represented the mathematical knowledge sub-domains of CCK (knowledge held in common with others who use mathematics in their work), SCK (mathematical knowledge specialised to the work of teaching) and KCS (knowledge of mathematics and of students). An overview of items on the form by sub-domain and by curriculum strand is presented in Table 4.1.

Table 4.1
Breakdown of survey items, by curriculum strand and by sub-domain

	Number & operations	Patterns, functions & algebra	Geometry*	Total
SCK	10	5	-	15
CCK	15	8	-	23
KCS	18	-	-	18
Geometry*	-	-	28	28
	43	13	28	84

*Note: Geometry items have not been classified into SCK, CCK and KCS

4.1.ii Schools from which Teachers were Selected

A random representative sample of schools was selected from Ireland's 3293 primary schools.[19] Each school was treated as a cluster and all teachers in the chosen schools made up the sample of teachers eligible to participate in the national survey of Irish teachers' mathematical knowledge. Schools were stratified by type and geographical region. The school types were disadvantaged, Gaeltacht, Gaelscoil, ordinary and special schools; and the geographical regions were Dublin, Leinster excluding Dublin, Munster, and Connacht/Ulster. Special schools were excluded as clusters from the study because they enrol pupils of both primary and post-primary age. Teachers of special classes in mainstream primary schools were included in the study. A random sample of schools was drawn from each stratum[20] – 87 schools in total (see Figure 4.1). The number of schools in each stratum is contained in Appendix 2. This resulted in a total possible sample of 670 teachers.

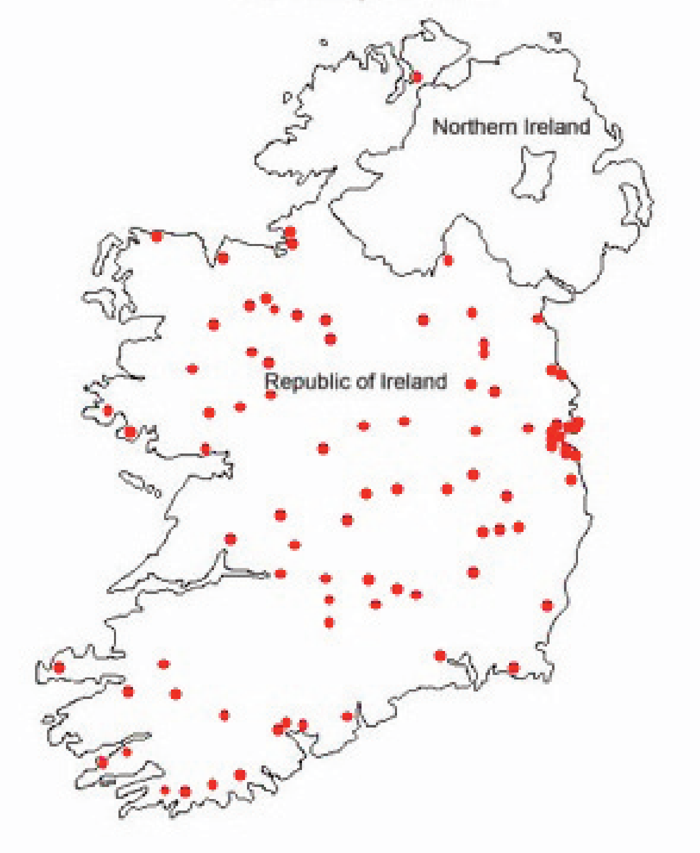


Figure 4.1
Approximate location of the 87 schools which were selected at random from primary schools in the Republic of Ireland to take part in the study.

4.1.iii Administration of Survey

Surveys were administered between June and December 2006. To maximise the response and to ensure consistency of administration, surveys were completed in the presence of the author or in the presence of an assistant survey administrator. The assistant survey administrators were either retired school principals or practising teacher educators. Schools were contacted by phone, by a follow-up letter and in some cases by visiting the school to ask if the teachers would take part in the study.

4.1.iv Response Rate

Almost without exception principals were supportive of the study and did their best to facilitate teachers in participating. Of the 670 teachers in the sample, 75% (n = 501) completed the survey. In 83% (n=72) of the schools, at least one teacher completed the survey. In schools where at least one teacher completed the survey, the average school

[19] Based on a list downloaded from www.education.ie on May 12, 2006.
[20] This was done by Brady West using PROC SURVEYSELECT in SAS software at the Center for Statistical Consultation and Research at the University of Michigan.

participation rate was 86% and 42 schools had a 100% response rate. At least six additional teachers agreed to take part but no convenient time could be found to administer the survey.

The response rate of 75% is high considering that teachers were asked to give up between 60 and 90 minutes to do what many teachers considered to be a mathematics test, in the relatively formal setting of having a researcher present. The strong response can be attributed to at least three factors. Many Irish teachers are favorably disposed towards educational research either because they have been involved in it in some way or they believe that it will benefit pupils. Many principals said this when I spoke to them and they encouraged staff members to participate. A second factor in the relatively high response rate is that the nature of the research meant that every school was contacted at least twice by phone and once in writing and many schools were contacted more than that. When teachers in a school agreed to participate, a venue and time for completing the questionnaire were agreed and the researcher was present to collect the forms at that time. Moreover, many schools were visited in person to ask the principal and/or the teachers if they would participate in the study. This direct contact contributed to the high response rate. The third factor is that every teacher who participated in the study received a nominal token of appreciation.

4.1.v Demographics of respondents

Demographic details of respondents were collected. In the final sample 84% of respondents (n=423) were female and 15% (n=75) were male. Three did not state whether they were female or male. In the entire population there were 26,282 teachers on September 30, 2004 – 83% women (n=21,789) and 17% men (n=4,493) – so the respondents had a similar gender composition to the primary teaching population. English was the first language of 94% of respondents (n=470) and 4% (n=20) had Irish as their first language. Two respondents were raised bilingually and nine did not answer this question. More than half the participants had 11 or more years teaching experience (see Table 4.2). Institutions from which teachers received their teaching qualification are listed in Table 4.3. Noteworthy is the fact that 16% of teachers surveyed received their initial teacher education in institutions other than the six traditional Irish providers of teacher education (Carysfort, Church of Ireland College of Education, Coláiste Mhuire Marino, Froebel College, Mary Immaculate College, St. Patrick's College).

Table 4.2
The number and percentage of teachers in the study by years of teaching experience

Experience	Number of Teachers	Percentage of Teachers
1 Year*	46	9
2 to 5 years	112	22
6 to 10 years	77	15
11 to 20 years	70	14
21 or more years	191	38

*191 teachers completed the questionnaire between September and December 2006 and a small number of them would have just begun teaching in September 2006. Because there was no option for "less than one year" these teachers may have ticked the box corresponding to having one year's experience. Four teachers did not state how long they had been teaching and one form was completed by a student currently enrolled in one of the colleges of education but who was working as a substitute teacher in a school on the day the questionnaire was administered.

Table 4.3
Where participants in the study received their pre-service teacher education

	Number of Teachers*	Percentage of Teachers
Carysfort	63	13
Church of Ireland College of Education	7	1
Coláiste Mhuire Marino	26	5
Froebel College	29	6
Hibernia College	21	4
Mary Immaculate College	147	29
St. Patrick's College	140	28
Other	59	12

*11 teachers did not respond to this question.

4.2 Variation in Teachers' Performances on the Measures

Teachers' performances on the measures will be reported using Item Response Theory (IRT) scores and difficulty estimates of the items. These scores take into account the relative difficulties of the items and reflect the fact that some items are better than others at predicting a respondent's overall MKT proficiency (Bock et al., 1997). As mentioned earlier, the scale used has an average of 0 and a standard deviation of 1 and a score of -3 indicates a teacher who, based on the item scores, has a low level of MKT and a score of +3 indicates a teacher with a high level of MKT. The difficulties of individual items on the survey are also estimated on a scale from about -3 (very easy item) to +3 (a very difficult item). An average item has a difficulty of 0, which means that a person with average MKT proficiency has a 50% likelihood of answering the item correctly (Hambleton, Swaminathan, & Rogers, 1991).

Many Irish teachers performed well on the measures and 15% of them were placed one standard deviation or higher above the mean (see Table 4.4). Satisfaction with finding strong levels of MKT among some Irish teachers must be tempered, however, by the fact that substantial variation exists among teachers in terms of MKT. The variation can be illustrated by thinking of the scores in relation to raw scores on the measures. A teacher at +2 on the scale responded correctly to around 40% more survey items than a teacher at -1 on the scale. A more extreme example is that a teacher at +3 on the scale responded correctly to around 60% more items than a teacher at -2 on the scale. This is a substantial difference in how teachers responded to items on the questionnaire.

Table 4.4
Numbers of Irish teachers placed on levels of the MKT proficiency scale. Mean = 0.

	-3	-2	-1	0	1	2	3
Number of teachers	2	12	67	165	182	62	11

Another way of thinking about this is that primary school pupils are being taught by teachers who bring vastly different knowledge resources to their mathematics teaching. Many teachers have the kind of knowledge needed to hear and interpret pupils' tentative mathematical ideas, to use accurate definitions that are comprehensible to pupils, to link multiple representations of number concepts, to skillfully choose and sequence tasks and so on. These teachers are well equipped to manage rich mathematical instruction as envisaged by the 1999 primary mathematics curriculum. Other teachers, however, have only a smattering of such knowledge. Their lessons are likely to be sidetracked into mathematically unproductive work, to be peppered with errors and omissions, and to miss opportunities to develop pupil understanding. Such teachers are unlikely to have the kind of mathematical knowledge needed to model and encourage mathematical practices such as reasoning, integrating and connecting, and applying and problem solving (Government of Ireland, 1999a). Most teachers' scores are located away from the extremes of high and low MKT, but scores are distributed along the scale. Although factors other than teacher knowledge influence instruction, without the kind of mathematical knowledge measured by the items it would be difficult for teachers to coordinate the rich mathematical instruction associated with high MKT.

Rather than being a type of knowledge held in more or less similar amounts by every teacher to support their teaching, the variability of teachers' levels of MKT suggests that among Irish teachers, possessing such knowledge is a matter of chance rather than a given. Because the teachers were selected from a nationally representative sample of Irish schools, the data suggest that Ireland's structures of pre-service and in-service teacher education are not systematically equipping teachers with broadly similar levels of MKT. It is therefore difficult to determine what might be a professionally acceptable level of MKT for teachers to possess.

Some might respond by saying that substantial variation in teachers' MKT is to be expected and possibly even accepted, claiming that there will always be teachers who bring different areas and levels of talent to enhance their teaching. Nevertheless, possessing MKT is an important factor in providing pupils with opportunities to learn mathematics. Some variation among teachers will always exist but the extent of variation found among the teachers in the entire sample – over 60% difference in the number of items answered correctly – seems remarkable,[21] raising the question of how some teachers managed to acquire MKT and others did not. Teachers with high levels of MKT may have acquired it through reading, by reflecting on their teaching, or by applying other mathematical knowledge to the work of teaching or in some other way. No matter how they acquired it, this study suggests that Irish primary teachers possess very different levels of MKT as a resource to enhance their mathematics instruction.

On reflection, it should come as no surprise that the level of MKT held by Irish teachers varies. Internal and external factors help explain it. One external reason is that for several years prior to the late 1980s, researchers in education paid relatively little attention to the topic of teachers' subject matter knowledge and its importance as a

[21] This is remarkable because entry to teaching has always been competitive (Greaney et al., 1999) and entrants to teaching in Ireland have traditionally been in the top quartile of their age cohort in terms of Leaving Certificate results (e.g. Greaney, Burke, & McCann, 1987).

resource for teachers. This began to change after Shulman and his colleagues inspired its return to the research agenda (Shulman, 1986; Wilson, Shulman, & Richert, 1987). From the early 1990s, there has been a lively interest internationally in studying teachers' subject matter knowledge, especially but not exclusively in mathematics, (Ball, 1990; Borko et al., 1992; Grossman, Wilson, & Shulman, 1989) and this research is now bearing fruit by linking what teachers need to know with the work they do and describing the knowledge teachers need (Ball & Bass, 2003). In this sense the lack of attention historically paid to teacher knowledge in Ireland is not exceptional and it contributes to explaining variability in teachers' MKT.

Factors internal to Ireland help explain the variation as well. Ireland's teachers have become more diverse in the last 10 years with teachers certified in other countries[22] and graduates from a new provider of teacher education joining the work-force. Furthermore, prospective teachers are not required to study mathematics as part of their teacher preparation program and most prospective teachers study no mathematics after completing secondary school. Moreover, recent in-service education for teachers has focused on conveying teaching methods rather than subject matter knowledge to teachers (Delaney, 2005). As a result, teachers are left to acquire what MKT they can, wherever they can. Research at the University of Michigan has contributed to an awareness of the complexity of the mathematical work of teaching mathematics and the benefits of taking seriously teachers' MKT. It seems timely that the type of mathematical knowledge teachers need and how they can acquire it be considered in Ireland.

4.3 Item Difficulties

Another way to consider the findings of teachers' performances on the MKT measures is in relation to categories of items that teachers found easy and difficult. Irish teachers found more survey items easy than difficult. As previously mentioned, each item was placed on a scale based on how teachers responded to the item; the scale corresponds to the teacher proficiency scale. Items with a difficulty level of -3 are very easy because a teacher with low MKT has a 50% chance of answering them correctly. In contrast an item at +3 is very difficult because even a teacher with high MKT has only a 50% chance of responding correctly. An item of average difficulty will be placed at 0 on the scale. Almost three quarters of the items (61 out of 84) had a difficulty level lower than zero, indicating that on average Irish teachers found more items easy than difficult. Figure 4.2 shows how items were distributed among different topics on the form according to difficulty. The average item difficulty level was close to -1 which means that a teacher with an MKT proficiency level of approximately -1 on the scale had a 50% probability of answering the average item on the survey correctly.

	N & O SCK	N & O CCK	N & O KCS	ALG CCK	ALG SCK	GEO
DIFFICULT						
		*				
		*				
						*
2			*			
		*				
	*		*			
			**			**
1		*				**
	**				*	
			*	*		
						***
0						
	*		*			
		**	*			***
		*	*	*	*	***
-1	*	*	**	*	**	**
		**	**		*	**
		**	*	*		
			*			***
-2	*	*				****
	*	*	**			
				**		*
				*		
-3		*	*			
	*					*
	*					
-4						
			*			
-5	*					
EASY						

Figure 4.2
The distribution of items by type (number and operations – N & O; algebra – ALG; geometry – GEO; SCK; CCK) and difficulty (mean = 0). * = item. The item difficulty estimates are on the left of the figure.

Areas that Irish teachers found less difficult, indicating strength in performance in those areas, were identifying and classifying mistakes made by pupils (with one exception, which will be discussed later), understanding of graphical representations of fractions, and algebra generally. The numbers of items related to the areas of strength are listed in Table 4.5a. Areas that Irish teachers found more difficult were

[22] This claim is based on data provided by respondents to the questionnaire and on the large numbers of teachers who sat the Scrúdú le hAghaidh Cáilíochta sa Ghaeilge in recent years (e.g. 533 in April 2007). This is an Irish language exam for teachers certified outside the state who wish to achieve recognition to teach in Ireland. Source: http://www.scgweb.ie (accessed on February 24, 2008).

applying definitions and properties of shapes, identifying and applying properties of numbers and operations, attending to and evaluating explanations, and linking number and word problems. Items with a difficulty level of 1.0 or higher (on the -3 to +3 scale) were considered to be difficult. The numbers of items related to each category are listed in Table 4.5b.

Table 4.5a
Areas of strength in Irish teachers' MKT

Area of Strength	Domain of MKT	Number of Items
Identifying and classifying pupil mistakes	KCS	3 (+1 exception)
Graphical representations of fractions	SCK	5
	CCK	1
Algebra	Algebra	4

Table 4.5b
Areas for potential development in Irish teachers' MKT

Area for Potential Development	Domain of MKT	Number of Items
Applying definitions and properties of shapes	Geometry	5
Identifying and applying properties of numbers and operations	CCK	3
Attending to and evaluating explanations	KCS	3
	SCK	1
Linking number and word problems	CCK	1

4.4 Areas of Strength in Irish Teachers' Mathematical Knowledge for Teaching

4.4.i Identifying and Classifying Pupils' Mistakes

Irish teachers generally know how to identify and classify pupils' mistakes. The item shown in Figure 4.3 is a typical example. The pupils portrayed in the item made three mistakes when applying a conventional subtraction algorithm. Most adults just need to be able to do the subtraction. A teacher has to do more: check if the pupil has answered correctly or not; identify any mistake; determine what may have caused the mistake; and in this particular teaching task decide which two errors are similar so that specific pupils can be supported in eliminating the type of error made. Teachers who possess the knowledge to identify errors have been found to be confident enough to allow pupils to make mistakes, and pupils have no reason to be afraid of getting a wrong answer (Schleppenbach, Flevares, Sims, & Perry, 2007). Teachers who are competent at identifying and classifying errors, as Irish teachers are, have the MKT that would enable them to use pupils' errors as resources to promote mathematical thinking in their classrooms and to plan further teaching keeping the likelihood of such errors in mind (Schleppenbach et al., 2007).

One exception to the overall strength in identifying and classifying errors was an item where teachers were required to diagnose the cause of an error. Specifically, teachers found it difficult to explain why a pupil might respond incorrectly to a maths problem of the form a + b = __ + d. Primary school pupils frequently respond to questions of this form by computing either one or both of the following sums a + b + d or a + b (Falkner, Levi, & Carpenter, 1999). If teachers know that pupils frequently interpret the equals sign as an order to compute rather than as an indicator of equality, teachers can plan their teaching to challenge the misconception. This area of teacher knowledge draws on knowledge of both mathematics content and students (KCS) and is related to identifying and classifying errors because it is knowledge teachers use when they respond to pupil errors.

Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students around particular difficulties that they are having with subtracting from large whole numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

I)	II)	III)
$\overset{4}{\cancel{5}}0\overset{12}{\cancel{2}}$	$3\overset{4}{\cancel{5}}00\overset{15}{\cancel{5}}$	$\overset{6}{\cancel{7}}\overset{9}{\cancel{0}}\overset{8}{\cancel{0}}\overset{15}{\cancel{5}}$
− 6	− 6	− 7
406	34009	6988

Which have the same kind of error? (Mark ONE answer.)

a I and II

b I and III

c II and III

d I, II, and III

Figure 4.3
Sample multiple-choice item developed by the Learning Mathematics for Teaching research team at the University of Michigan. Original item is released and available at http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

4.4.ii Graphical Representations of Fractions

Teachers in the Irish study performed well on problems where they were required to work with graphical representations of fractions. The representations included what Ni (2001) classifies as regional area models, a set model, a line segment and number lines (See Figure 4.4). Pupils’ learning of several fraction concepts, including that of equivalence, adding, and subtracting, can be enhanced when teachers use their knowledge of representations and translate between them (Bright, Behr, Post, & Wachsmuth, 1988). Irish teachers need to use their knowledge to make these translations because area models of fractions are the dominant form of representing fractions in Irish textbooks (Delaney, Charalambous, Hsu, & Mesa, 2007) and few problems require pupils to work across representations. The findings of this study show that teachers have the knowledge necessary to compensate for this shortcoming in textbooks. In another context involving graphical representation of fractions Irish teachers had little difficulty solving what Saxe and his colleagues (e.g. 2005) call an unequal area problem (see figure 4.5), which required respondents to identify a fractional part of a square partitioned in unequal parts.

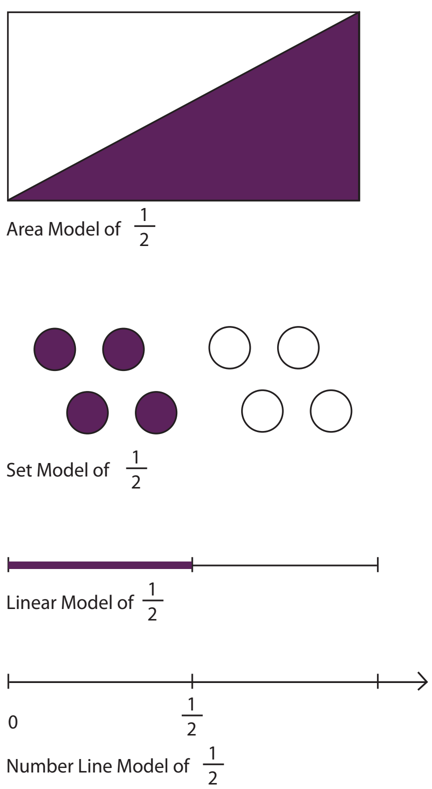


Figure 4.4
Graphical representations of fractions.

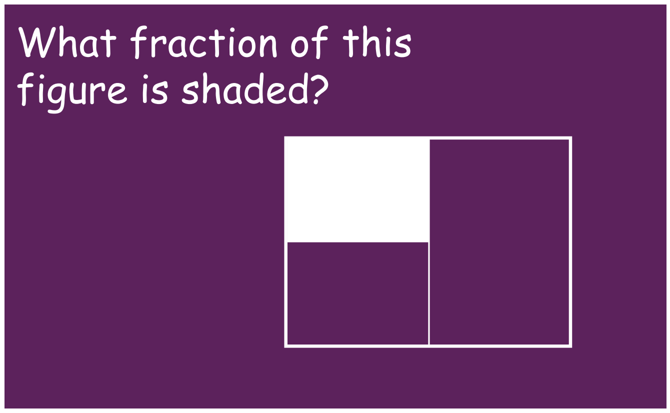


Figure 4.5
Unequal Area Problem.

4.4.iii Algebra

Another positive finding was that Irish teachers performed well on algebra. This is good because primary pupils generally find it difficult to make the transition from arithmetical thinking to the “relational thinking” required in algebra – thinking where pupils notice “number relations among and within” number equations and expressions (Jacobs, Franke, Carpenter, Levi, & Battey, 2007, p. 260). Relational thinking represents a more mathematically sophisticated way for pupils to understand arithmetic. If teachers can use their knowledge to help pupils make that transition in their thinking, pupils’ understanding of arithmetic improves and a strong foundation is laid for their subsequent understanding of algebra (Jacobs et al., 2007). From the evidence of this study, Irish teachers have the knowledge resources to do this.

Although the evidence from the teacher responses to this study show that Irish teachers are well placed to improve the teaching of algebra, a priority identified by Department of Education and Science Inspectors in the most recent National Assessment of Mathematics Achievement (Shiel, Surgeoner, Close, & Millar, 2006), a possible caveat must be mentioned. One survey question involved studying a pattern of 4 shapes repeated once, and required respondents to state what the 83rd shape would be. One way to do this algebraically would be to recognise that every whole number can be written in one of the following forms: $4n + 1$, $4n + 2$, $4n + 3$ or $4n + 0$ where n is a whole number. When one identifies the relevant form of a given number, it is possible to tell if the shape in that position of the sequence will be the first, second, third or fourth shape in the opening pattern. Solving the problem this way works for all numbers. It is also possible, however, to answer the question without using algebraic thinking, and judging by the annotations on some returned survey forms, at least nine teachers solved this problem by counting up to 83 in some form, such as writing 8, 12, 16, 20, 24, etc. below the shapes. This will work for finding the 83rd term but for numbers over a few hundred it would be a cumbersome way to find the answer and it does not involve the relational thinking mentioned earlier. It is difficult to know how widespread the arithmetic approach to the algebra item was among Irish teachers but it is an instance where the responses may not tell the full story about teachers' knowledge. Nevertheless, the survey responses indicate that over several items, Irish teachers performed well on algebra.

4.5 Areas for Potential Development in Irish Teachers' Mathematical Knowledge for Teaching

4.5.i Applying Definitions and Properties of Shapes

The set of geometry (shape and space) items was more difficult for Irish teachers than the algebra items. Item difficulties ranged from -3 to +3 but the average difficulty was -0.64. The 2004 National Assessment of Mathematics Achievement found that achievement of fourth class pupils was significantly better than it had been 5 years earlier and Department of Education and Science inspectors were more satisfied with how geometry was taught than in the previous study. Teachers, however, singled out geometry as an area in which they were less satisfied with the in-career development compared to their satisfaction with the treatment of number (Shiel et al., 2006). Perhaps the spread of geometry item difficulties in this study (-3 to +3) sheds some light on that finding. Irish teachers have strong knowledge in some areas of geometry, possibly contributing to good teaching (as noted by inspectors) and higher pupil achievement in these topics. Teachers seem to have less MKT in other areas and perhaps these topics were not addressed in professional development, contributing to some teacher dissatisfaction.

Teachers found it easy to identify one parallelogram in a series of two-dimensional figures, some of which were and some were not parallelograms. The easiest to recognise parallelogram, making it the easiest geometry item of all, was the one shown in Figure 4.6. It is not surprising that most Irish teachers recognised this figure because it is the example of a parallelogram typically given in Irish text books (e.g. Barry, Manning, O'Neill, & Roche, 2002; Gaynor, 2002). But recognising this shape does not indicate if the teacher has the knowledge resources to compensate for inadequate definitions of parallelograms presented in textbooks which frequently refer to rectangles pushed out of shape (Barry et al., 2002; Gaynor, 2002). Such definitions are inadequate because they do not help pupils or teachers to recognise that squares, rectangles and rhombuses, all being quadrilaterals with both pairs of opposite sides parallel, are all parallelograms. One instructional behavior associated with high MKT is careful use of definitions and in some cases MKT is needed to compensate for inadequate or inaccurate textbook definitions.

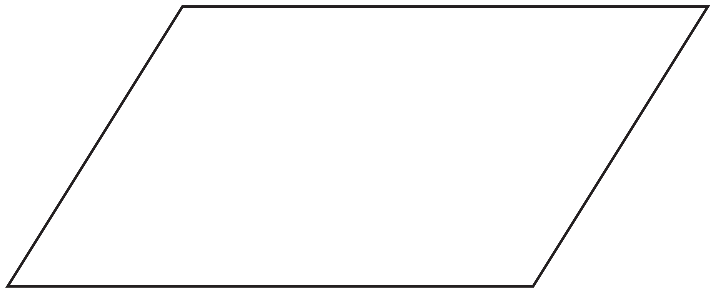


Figure 4.6
Irish teachers found this image of a parallelogram easy to identify.

Evidence in this study suggests that Irish teachers have difficulties applying definitions of shapes and shape properties. For example, the relationship between a square and a rectangle was problematic with most teachers seeing them as distinct shapes. Mathematically, a square is a special case of a rectangle where all sides are of equal length. Indeed, a square is a special case of a parallelogram, a quadrilateral, a trapezoid, and a kite (Weisstein, 2008). Classifying shapes in multiple ways makes demands on teachers' knowledge, in particular their knowledge of definitions and properties of shapes. For simplicity, many textbooks introduce shapes discretely, often with inadequate or no definitions. A related issue is that textbooks often present stereotypical examples of shapes, such as using illustrations of a regular hexagon and not qualifying it with reference to its regular quality. Such simplification may initially help pupils learn shape properties but it quickly becomes inadequate when pupils begin to investigate relationships among shapes, or test their understanding of shapes with non-examples or with non-standard examples. Teachers' mathematical knowledge is a necessary resource to prevent pupils acquiring misconceptions about shapes and to support pupils who become confused about whether a shape belongs or does not belong in a specific category. It is an area of MKT that many Irish teachers need to acquire.

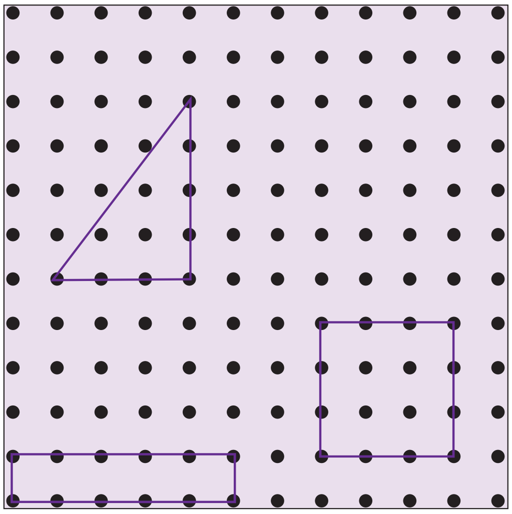


Figure 4.7
A pupil’s attempts to make figures with perimeters of 12cm on a geoboard.

Knowledge of geometrical properties can be helpful when using materials in mathematics class. The Irish curriculum suggests using geoboards to teach topics such as two-dimensional shapes, symmetry, and square and rectangular numbers (Government of Ireland, 1999a). Geoboards can be used to teach perimeter and an item on this topic was difficult for Irish teachers. The context was a classroom where pupils had been asked to make shapes with perimeters of 12cm on geoboards with pegs spaced 1cm apart (See Figure 4.7). The teacher was checking the work and one pupil had made a right-angled triangle with sides of 3cm and 4cm. Although the length of the third side could not be figured out empirically, the Pythagorean Theorem[23] could be applied to determine that the side length was 5cm and therefore, the total perimeter was 12cm. Most Irish teachers, however, responded either that the perimeter does not equal 12cm or there was not enough information to determine the perimeter. Most teachers encounter the Pythagorean Theorem in secondary school so why did they not apply it when responding to the item? It may be because they had forgotten it or it may be because they did not recognise the situation as one where the theorem may be applied. Interviews with teachers about their responses would be needed to determine the actual reason. Whatever the reason, it is an example of knowledge that is not part of the primary school curriculum, but knowledge which is useful for a teacher to have when setting tasks for pupils relating to perimeter.

4.5.ii Identifying and Applying Properties of Numbers and Operations

Irish teachers had difficulty identifying and applying properties of operations and properties of numbers. Many teachers appeared to lack the knowledge needed to evaluate rules of thumb frequently given to pupils, such as not taking a larger number from a smaller number. This type of task can be illustrated with an example. A teacher may be asked to consider the rule of thumb that “the sum of two numbers always results in a number that is bigger than both numbers.” If this rule of thumb is applied to counting numbers (i.e. 1, 2, 3, 4, 5…), it is clearly true. The smallest counting number is 1 and if one adds the two smallest counting numbers possible, 1 + 1, the sum is 2, a bigger number (See Figure 4.8).

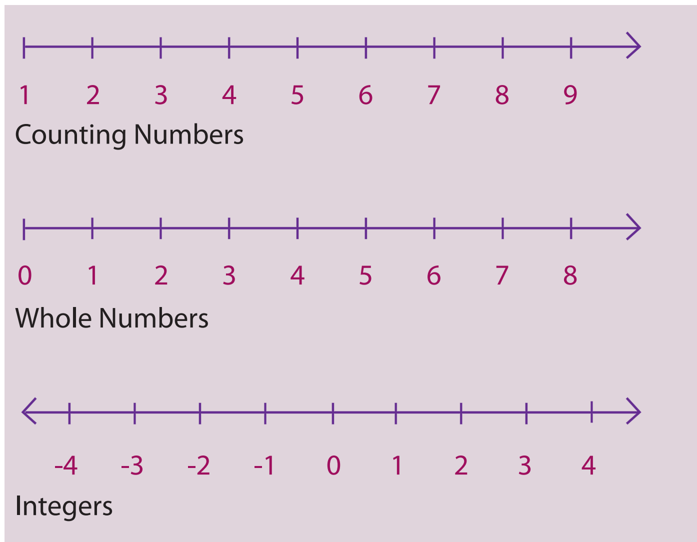


Figure 4.8
Three sets of numbers that are used in primary school mathematics.

But if the rule of thumb is applied to whole numbers (0, 1, 2, 3, 4, 5…), it is no longer true. Adding 0 + 0 equals 0 and this is not a bigger number. The sum of 5 + 0 is 5 and this number is not bigger than 5. If the numbers are extended to integers, the rule is untrue because adding -3 and -4 is -7 and -7 is smaller than both -3 and -4. Therefore, despite the intuitive logic that adding produces a bigger number, as a rule of thumb it is not always mathematically true. If pupils internalise such a rule, it may cause problems when they work with negative numbers in fifth and sixth class because they may think that say, -7 is greater than -3.

One reason why Irish teachers may have had problems evaluating properties of numbers and operations is that the teachers may have restricted the numbers they considered to counting numbers, which is the first set of numbers introduced in primary school. This is likely because a third of teachers agreed that it is always true that a larger

[23] The Pythagorean Theorem states that in any right angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides

number cannot be taken from a smaller number. These teachers know about integers from their study of secondary school mathematics and possibly even from teaching the topic in fifth or sixth class. In addition, a couple of teachers annotated their answers with comments such as "For whole numbers?" or "Are we talking about whole numbers or fractions?" Knowing the subset of numbers being referred to is part of the subject matter knowledge of teaching (Leinhardt & Smith, 1985). Another reason why these items were difficult for Irish teachers may be that they are not familiar with choosing key numbers on which to test such rules. For example, choosing numbers such as 0, 1, fractions or negative numbers can be useful for evaluating whether rules apply to numbers generally. Knowing properties and rules in relation to different sets of numbers and being able to choose useful examples for testing properties is important for primary teachers because by the end of primary school pupils have encountered whole numbers, integers, rational numbers and probably at least one irrational number (π). If pupils find that rules they were taught in younger classes no longer make sense as they move through the school, they may perceive mathematics to be a subject with arbitrary and incomplete rules. Such a perception is unlikely to contribute to pupils' understanding or to provide a strong foundation for future learning. A teacher who knows number and operation properties and who is clear about the number sets to which particular rules apply, is well placed to prevent pupils acquiring such misplaced ideas about mathematics. Such a teacher can be comfortable discussing with pupils when and why mathematical rules and properties apply, making the pupils more mathematically discriminating, opening up for them a vista of the mathematical horizon (Ball, 1993).

4.5.iii Attending to Explanations and Evaluating Understanding

The next area Irish teachers found difficult was in attending to pupil explanations and evaluating their understanding. The *Primary School Curriculum: Mathematics* (Government of Ireland, 1999a) document refers only a handful of times to the practice of explaining. Despite this, the video records revealed that several teachers requested and followed explanations from pupils in the video study. Attending to explanations and evaluating understanding may be difficult because many teachers have learned mathematics procedurally in school. Further, given the complexity of the tasks of communicating in mathematics class it should come as no surprise that attending to explanations and evaluating understanding is difficult for teachers generally. Irish teachers are no exception. When teachers were presented with pupils' explanations and asked to evaluate the explanations for evidence of understanding, they found it difficult. Figure 4.9 contains one problem that was difficult for Irish teachers. The item centres on a pattern on the 100-square which has the quality that anywhere a plus sign, three squares wide and three squares tall, is shaded, the sum of numbers on the row equals the sum of numbers on the column. Pupils were asked to explain why this is true for all similar signs. The task for teachers is to identify which explanation showed sufficient understanding of why the pattern is true for all similar plus signs.

Ms. Walker's class was working on finding patterns on the 100's chart. A student, LaShantee, noticed an interesting pattern. She said that if you draw a plus sign like the one shown below, the sum of the numbers in the vertical line of the plus sign equals the sum of the numbers in the horizontal line of the plus sign (ie., 22 + 32 + 42 = 31 + 32 + 33). Which of the following student explanations shows sufficient understanding of <u>why this is true</u> for all similar plus signs?
(Mark YES, NO or I'M NOT SURE for each one.)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

	Yes	No	I'm not sure
a) The average of the three vertical numbers equals the average of the three horizontal numbers	1	2	3
b) Both pieces of the plus sign add up to 96	1	2	3
c) No matter where the plus sign is, both pieces of the plus sign add up to three times the middle number.	1	2	3
d) The vertical numbers are 10 less and 10 more than the middle number	1	2	3

Figure 4.9
Sample multiple-choice item developed by the Learning Mathematics for Teaching research team at the University of Michigan. Original item is released and available at http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

Four pupil explanations are presented:

a. The first one states that in any plus sign shape on the 100-square the average of the three vertical numbers is the same as the average of the three horizontal numbers. If the averages of two equal-size sets of numbers are equal then it follows that the sums of both sets of numbers are equal. This response shows evidence of understanding why the pattern is true.
b. The second response simply makes a statement about the specific plus sign shaded on the 100-square. It gives the specific details that the vertical and the horizontal lines are equal by adding them. Nothing said explains why this might be true in other parts of the 100-square and the statement does not move much beyond the original pupil's claim.
c. The third pupil's explanation uses another relationship between the row and the column to explain why the pattern is true. The pupil notes that the three numbers on both row and column add up to three times the number in the middle. This observation, which is generalised to "no matter where the plus sign is", shows understanding of why the pattern holds in every case: if the three numbers add up to three times the middle number and the middle number is the same for the row and the column, the sums of the row and the column will be equal.
d. The fourth response shows insufficient understanding of why the pattern applies. The statement is true but it refers only to the numbers in the vertical column, not to the numbers in the horizontal row. In order to show understanding, an explanation must show a relationship that exists between the vertical and horizontal rows.

Irish teachers found the 100-square item difficult, especially parts (b) and (d) where they frequently accepted statements as showing understanding which did not meet the standards of understanding required. Other items requiring evaluation of pupil explanations were also difficult. Items included explanations of the decomposition algorithm for subtraction and why reducing fractions produces an equivalent fraction. The difficulties Irish teachers had with these items demonstrate that attending to a pupil explanation (orally or in writing) is difficult. The teacher needs to know what constitutes an adequate explanation of the particular mathematical idea; the teacher needs to be able to interpret what the pupil produces and compare the two before evaluating the pupil's understanding. A teacher uses mathematical knowledge to respond to the pupil or to ask for further elaboration. The teacher does not have time to check facts in a book and respond later. Even if a book is consulted, mathematical judgment will always be required because the form and content of pupil explanations are frequently unorthodox and rarely predictable. Being able to follow and evaluate a pupil's mathematical explanation draws on a teachers' knowledge of content and students (KCS).

4.5.iv Linking Number and Word Problems

Many studies of pre-service teachers have expressed concern about the depth of their understanding of arithmetic operations (e.g. Chapman, 2007), and this understanding can be particularly shallow when operations with fractions are involved (e.g. Borko et al., 1992). Most operations have multiple meanings or structures (Haylock, 2006), such as the equal sharing and repeated subtraction meanings of division. Teachers need to understand the meanings of operations when identifying the operation implicit in a word problem or when writing a word problem for students to work on. Irish teachers had difficulties matching a word problem to the fraction problem $^1/_2$ - $^1/_3$. The difficult aspect of writing and interpreting word problems based on fractions is the notion of what constitutes a whole. For example, a word problem such as, "Mary had a $^1/_2$ box of sweets and she gave $^1/_3$ of the sweets to her brother. What fraction of her sweets was left?" may at first glance appear to match the calculation. It mentions both numbers and the phrase "gave … sweets to her brother" implies subtraction. But a more detailed look at the question reveals that for the half, the implicit whole is the box of sweets; and for the third the implicit whole is the half box of sweets. Therefore, that word problem is not a good match for the number problem $^1/_2$ - $^1/_3$. The word problem as written would be solved using the numbers $^1/_2$ - $^1/_6$ where both fractions refer to the whole box of sweets.

Matching word problems and fraction calculations, and drawing attention to the relevant whole unit, is important for Irish teachers because the curriculum wants children to see mathematics as "*practical* and *relevant*" (Government of Ireland, 1999a, p. 15, italics in original) but popular Irish textbooks present no worked examples of fraction computations in practical contexts (Delaney et al., 2007). Matching word problems with calculations draws on teachers' CCK and it is knowledge that many Irish teachers do not currently hold.

4.6 Summary of Teachers' Performances

This report has shown that many Irish teachers scored highly on the survey items and on the whole Irish teachers found more MKT items easy than difficult. Among teachers generally, however, MKT varies widely. Teachers exhibited strong MKT across all algebra items. Performance on geometry (shape and space) ranged more widely than algebra but overall, teachers performed less well on this strand. Teachers had difficulties in applying properties and definitions of two-dimensional shapes. Teachers performed well on knowledge of different graphical representations of fractions. They had few problems identifying a fractional part of an unequal area shape but evaluating rules about number properties and operations, and matching a fraction calculation to a word problem (especially when the whole unit changes) was more difficult. Teachers had few problems identifying and classifying pupils' mistakes, but attending to explanations and evaluating pupils' understanding was problematic.

Several Irish teachers performed well on the measures of MKT in this study but many pupils are taught by teachers who responded incorrectly to several items. Details of shortcomings in teachers' mathematical knowledge have become apparent as more is learned about the specialised nature of what teachers need to know. Raising the mean and reducing the variation of knowledge held will require determined effort. The variation and difficulties in teachers' mathematical knowledge today are understandable because little was known about MKT generally or specifically about Irish teachers' MKT. With what is now known internationally and nationally, the opportunity exists for teachers, policy makers and teacher educators to develop MKT among all teachers and prospective teachers. If that is done, the quality of mathematical instruction is likely to improve, which should help raise student achievement in mathematics.

5.1 Summary

Much has been learned about mathematical knowledge for teaching (MKT) in the United States and in other countries over the last 20 years. This study shows that MKT, as elaborated in the United States, matters for teaching in Ireland, because the mathematical work of teaching observed in Irish lessons is similar to the work of teaching that informed MKT in the United States. Furthermore, teachers' scores on the measures are related to the mathematical quality of instruction observed in lessons taught by the teachers. Teachers who score well on the measures tend to exhibit higher quality mathematical instruction than teachers who score poorly.

When the MKT measures were administered to 501 primary teachers from a national sample of 72 Irish schools, the level of MKT varied substantially among teachers. This finding is important because it means that although many teachers have the knowledge resources to coordinate mathematical instruction of a high quality, many others do not. The extent of variation in mathematical knowledge that is related to instruction is disappointing, even if it is understandable. If it were discovered that knowledge essential to the work of engineering, nursing or plumbing was unevenly held among engineers, nurses or plumbers respectively, the public would be concerned. But at least in those jobs, collaboration is often a feature of the work; knowledge held by one worker may be complemented by knowledge held by a colleague or another team member. Teaching is more isolated, with teachers doing most of their work behind closed doors, with other teachers rarely present (Labaree, 2000). As a result of the variation in knowledge and the isolated nature of teaching, pupils are learning mathematics in classrooms where teachers bring vastly different resources of MKT to the work of teaching. Several reasons account for this. First, it is difficult, if not impossible, for teachers to have expert knowledge in all subject areas. Second, for many years the knowledge needed to teach mathematics at primary school level was underestimated by researchers, policymakers, teacher educators and even teachers themselves. Consequently, teachers were not expected to possess mathematical knowledge other than what they learned in primary and post-primary school. Third, the nature of the mathematical knowledge required was not specified so teacher educators at pre-service and in-service levels had no research base to inform the design of mathematics courses for teachers.

This study identified areas of strength and areas of difficulty in Irish teachers' mathematical knowledge for teaching. Teachers knew how to identify and classify pupils' mistakes; their knowledge of graphical representations of fractions was good, and so was their knowledge of algebra. Areas of difficulty included knowledge of applying definitions and properties of shapes, and properties of numbers and operations: teachers tended to over-generalise properties of counting numbers to all subsets of the number system. Attending to pupil explanations and evaluating pupil understanding was difficult for teachers, as was linking fraction number and word problems.

5.2 Goals for the Future

In order to respond to the findings of this study, two goals can be identified:

1. Address the variation in teachers' mathematical knowledge for teaching, by systematically developing it among practising and prospective teachers.
2. Prioritise support for all teachers in the areas of MKT that Irish teachers currently find difficult: applying definitions and properties, linking number and word problems in fractions, and following pupils' explanations and evaluating pupils' understanding.

No single initiative can realise these goals but the potential benefits of achieving them make a series of strategies worthwhile. One positive finding is that some teachers have acquired MKT despite the lack of a systematic approach to developing such knowledge.

5.3 Recommendations

Although poor Leaving Certificate results in mathematics attract media attention when they are published each year, Irish students' mathematical attainment is average when compared to students in other countries. Yet, given that Irish students' scores in reading literacy and science are significantly higher than OECD country average scores (Cosgrove et al., 2005; Eivers et al., 2007), improvement in Irish students' mathematics scores is both possible and desirable. Furthermore, *Ireland's strategy for Science, Technology and Innovation 2006-2013*[24] states that future success depends on "ensuring that levels of scientific and mathematical literacy increase." In this context the recommendations below are proposed for developing teachers' mathematical knowledge and consequently to raise student achievement by improving the mathematical quality of instruction in primary schools.

In order to ensure accountability for implementing the recommendations an individual or a committee should be appointed to oversee, monitor and rigorously evaluate each recommendation, within a pre-specified, realistic timeframe. Too often evaluation of professional development for teachers has been absent or weak. Rarely is objective information gathered, for example, on the effect of professional development on classroom practice or on

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[24] http://www.entemp.ie/publications/science/2006/sciencestrategy.pdf

student outcomes. The instruments used in this study may be used to evaluate future professional development in mathematics: to measure growth in teacher MKT and to study the mathematical quality of instruction. Information gathered by these instruments can help ensure that only those initiatives which are shown to work are continued. The specific recommendations are as follows:

- Design, deliver and evaluate professional development for teachers that is built around the practice of teaching.
- Use pupils' textbooks and ancillary materials as one way to develop and support teachers' MKT.
- Require all prospective teachers to study MKT as part of their initial teacher education programme.
- Investigate the feasibility and benefits of having specialist teachers of mathematics in some schools.
- Provide mathematics courses and accompanying discussion forums online.
- Raise the mathematics requirement for entry to teacher education.
- Support research into the relationship between teachers' mathematical knowledge and pupil attainment.

Each recommendation will now be described in more detail.

5.3.i Design, Deliver and Evaluate Professional Development for Teachers

The first initiative relates to professional development for teachers and the suggestions presented are influenced by ideas from Ball and Cohen (1999). In agreement with Ball and Bass (2003), this report concurs that teaching is mathematically demanding work which requires a special kind of mathematical knowledge. Teachers draw on this knowledge when they are teaching mathematics, often while simultaneously responding to several other teaching issues from timetable constraints to pupil misbehavior. Therefore, professional development needs to be connected closely to the practice of teaching mathematics. The topics identified above should be prioritised for attention: attending to pupil explanations and evaluating understanding, applying definitions and properties of shapes, applying properties of numbers and operations, linking fraction calculations and word problems, and interpreting alternative algorithms.

In order to keep these topics close to practice, sessions for teachers need to be designed around classroom practice. Two ways of doing this are using primary mathematics laboratories or a variation of Japanese Lesson Study.[25] A primary mathematics laboratory is where one teacher teaches a group of pupils over a period of time, say a week, and the teaching is observed and studied by other educators who attend a pre-briefing beforehand and a debriefing afterwards in which the planning and the execution of the lesson are discussed. Although laboratory schools were in use a century ago, the idea has been revived more recently at the University of Michigan; and weeklong summer courses in mathematics using the laboratory school model were organised in Coláiste Mhuire, Marino Institute of Education in 2007 and 2009. Japanese Lesson Study is similar in that it is centred on lesson observation and analysis. Although lesson study varies throughout Japan, the focus is more on improving one or several lessons by revising them and teaching the revised lesson based on the evaluation of the original lesson (Stigler & Hiebert, 1999). Observation of teaching is currently used in the dissemination of teaching practices associated with the Reading Recovery programme in Ireland.

An alternative to observing live teaching practice would be to collect records of classroom practice in which mathematical tasks of teaching arise (such as those identified above) and use them to stimulate teacher discussion. Time would be needed to collect useful examples of video records of practice, but the study summarised in this report shows that it is possible. A video record of a pupil explaining a mathematical idea could be used to stimulate discussion in a professional development session. Teachers could discuss in advance what an adequate mathematical explanation of the idea would be, and subsequently discuss elements that were present and absent in the pupil's explanation, and what evidence existed of pupil understanding or misunderstanding. The teachers could discuss what makes an explanation clear for the teacher and for other pupils, and what pupils would need to learn so that they could explain and follow explanations in this way. Teachers could relate their own instructional contexts to the pupil actions, the teaching actions and the mathematics observed in the video. Other records of practice such as pupils' work and teachers' notes could supplement the video records. In addition, teachers could complete and discuss primary school mathematics tasks.

Teachers could participate in such professional development sessions on a regular basis, say a half day every month. This might encourage and enable teachers to develop language for discussing practice and to engage in robust discussion of teaching and knowledge for teaching, getting beyond the politeness that characterises much discussion about other teachers' teaching. Leaders of such professional development need to possess high MKT, as well as knowledge of how teachers learn. In addition, specific preparation would need to be planned for such leaders so that they have opportunities to discuss the practice of teaching themselves before they lead teachers in such discussions. The model of trainers and cuiditheoirí, which was used by the Primary Curriculum Support Programme

[25] See http://lessonresearch.net for more information about Japanese Lesson Study. Site accessed on September 21st 2009.

(now the Primary Professional Development Service) could be adapted for developing teachers' MKT. Alternatively, one nominated person with high MKT and knowledge of teacher learning could be assigned to each education centre around the country; such a person could support teachers in developing MKT in the area served by the education centre.

In order to ensure prudent use of resources, the design and delivery of the professional development needs to be informed by research on teacher learning. Formal evaluation needs to be built into the programme to ensure that it achieves the goal of enhancing teachers' MKT. The multiple-choice measures used in this study would offer a concrete way to evaluate the impact of any professional development on teachers' knowledge.

Practical issues would need to be addressed in order for teachers to find the time to attend the professional development sessions, but current precedents point to some possibilities. Instead of closing a school for a full day to facilitate teacher attendance, schools might be closed for half days. Alternatively, teachers might attend the sessions in their own time – evenings or weekends – and receive time off in lieu as is the current practice for summer courses under DES Circular 37/97; teachers might take one day's EPV leave for every three half-day sessions attended, for example.

5.3.ii Use Pupils' Textbooks and Teachers' Manuals to Support and Develop Teachers' Mathematical Knowledge for Teaching

The second recommendation is that the Department of Education and Science take a proactive role in monitoring the quality of textbooks used in schools and that from a certain date only textbooks which meet the approval of the Department be used in Irish national schools. Pupils' textbooks are used by many teachers on a regular basis in their teaching and such materials could help teachers develop components of MKT, in relation to definitions and properties of shapes, for example. They could also play a role in supporting the teachers' existing MKT. Consideration might be given to having pupils' textbooks written by multidisciplinary teams consisting of teachers, mathematics teacher educators and mathematicians, each with expertise in the area of textbook development. Each of these perspectives, combined with relevant research, could improve the mathematical quality of textbooks and supporting teachers' manuals. Although teachers may be reluctant for textbooks to be changed radically, the idea of "replacement units" (e.g. Wilson, 2003) used in the United States, where the treatment of one topic at a time is revised, may be used initially in conjunction with existing textbooks to elicit feedback from teachers on this initiative. Ball and Cohen (1996) have written about the potential of curriculum materials in teacher learning.

5.3.iii Require all Students to Study Mathematics Content as Part of their Teacher Education Programme

Not all student teachers are currently required to study mathematics during their teacher preparation programmes, but requiring all students to take at least one mathematics content course is worth considering. The content of such a course needs to provide teachers with mathematical knowledge that is used in and useful for teaching. Mathematics teacher educators and mathematicians familiar with MKT could work collaboratively to design such courses and to monitor their success. The approach used with prospective teachers will differ somewhat to that used with practising teachers because the former have little or no teaching experience to which they can relate the knowledge demands of teaching. By having opportunities to discuss practice, however, they may acquire dispositions towards practice that will prepare them to develop MKT through reflection on their teaching when they begin working fulltime. Ideas and materials from research groups in other countries such as Mod4[26] should be helpful in developing MKT among prospective teachers.

5.3.iv Investigate the Practicality of Having Specialist Teachers of Mathematics in Some Schools

Another possibility to be considered is whether there is a role for specialist mathematics teachers in primary schools as currently exists in secondary schools. The question has practical dimensions as well as theoretical ones. Based on the evidence of the findings above, teachers have widely varying levels of MKT, and MKT levels are related to instruction. Based on U.S. research (Hill et al., 2005), it seems probable that the mathematical quality of instruction is associated with pupil achievement. Therefore, if Teacher A in a school has substantially higher MKT than Teacher B, more pupils would benefit from higher mathematical quality of instruction if Teacher A taught Teacher B's class for mathematics. In many schools, however, such an arrangement may be difficult to organise. It might work well in a large school, for example, if one teacher with high MKT taught mathematics to fifth and sixth classes, and another teacher taught, say, English to both class levels. In smaller schools such an arrangement may not be practical.

5.3.v Offer Mathematics Courses and a Discussion Forum Online

More and more professional development for teachers is now offered online and an online environment could be used to develop teachers' MKT. For example, a moderated discussion forum where teachers can discuss issues related to mathematical knowledge for teaching may be useful. On such a forum teachers could raise and respond to questions relating to mathematical knowledge. Such an initiative might follow or accompany the professional development initiative outlined above. As a stand-alone

[26] See http://sitemaker.umich.edu/mod4/home for more information. Accessed on September 21st 2009.

initiative, teachers might not have a context for the kind of collaborative discussion about the practice of teaching and mathematical knowledge that such an online forum could support. Without active teacher commitment to and participation in such a forum, it would serve little purpose.

An online environment could also be used to provide more formal courses in MKT for teachers. Such courses have the advantages that more teachers could participate in them than on traditional professional development courses and that they are accessible to teachers all over the country. Like the other initiatives listed above, it would be important to monitor teacher learning on such courses. This study has used multiple-choice measures that can be used to monitor any of the initiatives aimed at developing teachers' MKT.

5.3.vi Raise the Mathematics Requirement for Entry to Teacher Education

Other recommendations can also be considered but they have less of a basis in research. For example it has been recommended that the minimum Leaving Certificate mathematics entry requirement be raised (Department of Education and Science, 2002). Such a move may be of more symbolic than of concrete value, because at best it is likely to improve only the common content knowledge held by prospective teachers. But it may have the effect of recruiting into teaching more people who are confident and competent in their approach to studying mathematics.

5.3.vii Support research into the relationship between teachers' mathematical knowledge and pupil attainment

Much has been learned about teachers' mathematical knowledge over the past two decades. But in Ireland and elsewhere much remains to be learned. Because Ireland is the first country to conduct a national study of teachers' MKT, it is well placed to conduct further research in the area. Questions to be addressed in future research include:

- Is there a link between teachers' MKT and pupils' attainment in Ireland? Interest in teachers' knowledge is rooted in an aspiration to raise student achievement levels. Therefore, it would be helpful to see if students taught by teachers who possess more MKT, achieve higher gains in their scores on standardised tests.

- How do teachers' and prospective teachers acquire MKT? Frequently, professional development has been designed and offered to teachers based on intuition, rather than on research on teacher learning. Therefore, any serious attempt to raise teachers' knowledge needs to be grounded in available research, and to contribute to such research. Initiatives can only be considered successful if they help teachers to raise their levels of MKT.

- Apart from what has been learned about MKT in the United States, what additional elements of MKT do Irish teachers know and need to know? Much has been learned about the mathematical work of primary teaching in the United States and its knowledge demands. By studying the mathematical work of teaching in Ireland, more can be learned about the knowledge needed to do the work.

- What MKT is used and needed by teachers of early childhood classes? Some readers may question if all primary teachers need to have high levels of MKT; specifically, do teachers of junior classes really need the same MKT as teachers of senior classes? It must be acknowledged that the mathematical demands of teaching junior primary school classes have not yet been comprehensively documented, so more research is needed on the work of teaching at this level. Nevertheless, when Hill and her colleagues (2005) studied gains made by pupils in their scores on standardised mathematics tests, they found that teacher knowledge, as measured by items similar to those used in the study reported here, made a difference in the achievement of first grade pupils, the youngest age group studied. This finding from the United States suggests that teachers' levels of MKT make a difference in the achievement of young pupils, even if the topics and tasks of teaching contained in the items appear to relate to more senior class levels.

- What mathematical knowledge for teaching is used and needed by post-primary teachers? Work has begun on studying teacher knowledge at middle school[27] level in the United States. Studying the mathematical knowledge used by secondary teachers would inform the preparation and professional development of post-primary school mathematics teachers. Ireland can contribute to the developing work of understanding the knowledge needed for teaching by studying the mathematical work of teaching at all class levels.

5.4 Conclusion

It would be an oversimplification of the complex work of teaching to claim that increasing teachers' mathematical knowledge alone will lead directly to improved instruction. Lampert (2001) compares teaching to "navigating an unwieldy ship on a large and tumultuous body of water" (p. 446). She goes on to say that

> With the appropriate tools and knowledge, you can usually determine where you are, where you need to go, and where everyone else is in relation to where they need to go, but not always (p. 446).

That quotation conveys a compelling metaphor for

[27] Middle school in the United States equates to sixth class in primary school and first and second year of secondary school in Ireland.

mathematics teaching and for the role teacher knowledge plays in it. Subject matter knowledge is part of the knowledge that usually helps practice, but not always. More research is needed on why knowledge does not always help practice.

Concerns exist about Irish students' mathematical achievement in post-primary school mathematics. Primary school provides the foundation on which students build their post-primary mathematics learning and where they acquire dispositions towards the subject. This report finds that levels of knowledge vary among Irish teachers and it identifies areas of strength and weakness in the knowledge currently held by teachers. Evidence is provided that teachers' knowledge matters for teaching. Developing teachers' mathematical knowledge has the potential to help teachers find the teaching of mathematics more stimulating and professionally fulfilling. Furthermore, it offers one concrete way for teachers to provide higher quality mathematics instruction, leading to higher student achievement in mathematics.

Afterword

Studying the work of teaching mathematics to determine the knowledge teachers need has produced insights into the teaching of mathematics, but in Ireland such an approach may also be helpful in developing teachers' knowledge of the Irish language, another priority area for the Department of Education and Science.[28] It is likely that the proficiency teachers need in speaking Irish differs from the language proficiency needed by say, an author, a broadcaster, a translator or a historian working through the medium of Irish. Teachers need to be able to select vocabulary that provides learners with high leverage in speaking the language early on; they need to anticipate common errors students make; they need to know how to express common classrooms phrases accurately in Irish; they need to be able to present rules in understandable ways; they need to be able to select contexts in which the language can be practiced and so on. Just like MKT, this work seems to require a special type of knowledge of the Irish language, over and above language teaching methods. The specific type of language needed could be studied by carrying out a form of task analysis of the work of teaching the Irish language, similar to the analysis done for mathematics by Ball and Bass, and it could yield fruitful results for understanding the Irish language knowledge that is needed for teaching the subject.

[28] See, for example, this press release from April 2006 which lists three initiatives aimed at developing teachers' language fluency: http://www.education.ie/home/home.jsp?maincat=&pcategory=10861&ecategory=40280&sectionpage=12251&language=EN&link=link001&page=20&doc=30795 Accessed on April 8, 2008.

Appendix 1

Tasks of mathematics teaching identified in ten Irish lessons, with descriptions of the tasks and indications of the mathematical knowledge needed to do the tasks

Task of Teaching	Description of Task (as it could happen but may have happened differently in example)	Sample MKT Demands
Connect a mathematics problem to a skill for living	Teacher relates a mathematics problem to an activity related to life outside school (e.g. managing a budget).	• Know how mathematics can be applied in society for a citizen's benefit
Apply mathematics in the pupils' environment (in school and out of school)	Teacher uses mathematical examples from the pupils' environment. For example, the teacher points out that an item in the pupils' environment is an example of a particular shape, or property of a shape or that it is a quantity of a particular size (length, capacity, area etc.) Alternatively, the teacher must decide if what a pupil calls an example of a shape (or property of a shape) is a correct example. Teacher asks questions which pupils can answer using information given in class but where the context in which the information must be applied is different.	• Know names and properties of shapes in the curriculum Determine if a particular shape fits a category (e.g. if one pupil suggests a door as a rectangle and another suggests it as a rectangular prism) Know benchmarks for common measures Recognise contexts where pupils can apply mathematics taught in class.
Tell pupils what they will be working on in a lesson	Teacher outlines what pupils will be working on in the lesson. It could also occur if there is a transition in a lesson or if the problem type changes.	• Know how to present a topic in a way that will be comprehended by pupils and that will stimulate their interest
Tell pupils why they are doing an activity	Teacher states specifically what pupils will learn or practice by doing a particular task or game in class.	• Know the mathematical purpose behind tasks sourced in textbooks, teacher courses, from other teachers, from the internet.
Tell pupils what they have been doing in a lesson or activity	Teacher tells pupils what they have learned without stating the key points again.	• Know what the mathematical focus of the lesson is
Identify salient information in a lesson or topic	Teacher points out to pupils the most important aspects of a topic (a shape, a definition, an algorithm etc.) to which they should attend	• Know the key points in a given topic Know which aspects of a topic will help future learning
Decide not to pursue a topic in a lesson	Teacher decides not to pursue a topic that is introduced by a pupil.	• Know which aspects of a topic will be productive in terms of mathematics learning and which will not

Choose numerical or geometric examples for the lesson	Teacher chooses examples relevant to what is being taught and that work. Examples need to be appropriate for the age and stage of the children. For example, to teach division with no remainder the numbers 78÷9 would not be good, or to teach subtraction without regrouping 72 – 24 would not work. Similarly an equilateral triangle would not be good if you wanted to study right-angled triangles, or a circle if you wanted to study polygons.	• How to calculate the answers to the examples • Know the range of numeric or geometric examples that are available for selection
Connect current topic to material pupils will work on in the future	A teacher explicitly relates content being taught to something pupils will learn in the future, most probably not in the current class level. Casual references to something that will be done or finished tomorrow are not included.	• How “rules can change”: e.g. in second grade saying that one cannot take 6 from 2 and expecting pupils’ to do that in sixth grade. • Continuum of a topic
Connect current lesson topic with material learned in a previous class level	A teacher explicitly relates a topic to something studied in a previous class level.	• Knowledge of the mathematics curriculum outside the current class taught • Links between and among different mathematics topics (e.g. fractions and decimals) • Continuum of one topic (e.g. multiplication of fractions and division of fractions)
Connect current lesson topic with work done in a previous lesson in current class level	A teacher explicitly relates a topic to a previous lesson done any time before in the current class level.	• How a topic is sequenced • How a topic links to other topics • How a previous topic can help or hinder understanding of a new topic (e.g. decimals and money)
Ask a mathematical question on a topic not taught in the lesson (but which at least some pupils are expected to know)	The teacher asks the pupils a mathematical question on a topic that does not feature in the current lesson. Teacher asks pupils to compute using an operation not in the lesson.	• Know what prior mathematical knowledge pupils can be expected to have • Know how pupils’ prior mathematical knowledge can be incorporated into a new topic
Recap on mathematics practiced so far in lesson	The teacher summarises what mathematics has been done so far in the lesson.	• Know what the key points of the lesson are
Ask questions to revise material in lesson	Teacher asks questions to revise material covered in the lesson. The answers require repetition of mathematics information already presented in class. [Excludes questions asked to elicit names or properties of shapes]	• Know what the key points of the lesson are

Respond to a mathematical question from a pupil	Teacher responds to a mathematical question that pupils ask.	• Know what the pupil is trying to understand • Frame an answer in a way that is comprehensible to the pupil • Know what resources can be accessed to assist in responding to questions when the teacher does not know the answer
Help or prompt a pupil who is stuck or incorrect (e.g. giving a clue or a suggestion)	Teacher responds to a pupil who has an incorrect answer or who is not making progress on work by giving some form of support. It might be in the form of a question or a clue as to the answer or to change the context of the problem. This refers to a short, focused one or two sentence intervention.	• Identify what caused the pupil's error or what is preventing the child from continuing to work • Know what question or clue could be most productive in advancing the pupil's mathematics learning
Explain mathematical ideas	Teacher explains a mathematical idea to pupils using words, pictures, examples or other materials. This task is distinct from a teacher eliciting an explanation from or co-constructing an explanation with pupils.	• Understand the idea • Know the key parts of the idea (including required background knowledge) and sequence them appropriately • Know how to communicate the idea to elementary pupils
Help a pupil describe a mathematical procedure	Teacher helps the pupil to construct the description by questioning the pupil or by supplementing the description with necessary information.	• Understand what the pupil is describing • Have the knowledge necessary to supplement any relevant information omitted by the pupil
Anticipate ideas that may be confused by pupils	Teacher anticipates what may cause difficulties for pupils when teaching a topic. This may be done by pointing out common errors, highlighting important differences or by giving pupils time to understand one idea before introducing the second.	• Know common pupil errors (e.g. that if the ones digit in the minuend is less than the ones digit in the subtrahend, pupils are likely to take the minuend from the subtrahend)
Elicit the meaning of an operation	The teacher asks questions so that pupils will state at least one meaning of a number operation.	• Know the different meanings of number operations (e.g. regrouping and equal addition for subtraction; partitive and measurement for division)
Teach pupils how to write numerals or other mathematical symbols	Teacher gives pupils specific guidance on how to write numerals for small and large numbers or other mathematical notation. Examples might include teaching young pupils a single numeral or differentiating algebraic x and multiplication sign for older pupils.	• Know how numerals/symbols are written • Knowing difficulties pupils are likely to have in writing numerals or symbols

Ask pupils how numerals or mathematical symbols should be written.	Teacher does not ask pupils to write but asks how certain numbers would be written. For example large numbers (say ten thousand and fifty or half a litre as a decimal)	• Know what numerals pupils find difficult to write. • Know conventional forms of saying numbers (e.g. “Three point twenty,” or “three and twenty hundredths”)
Write numerals and operation signs on the board	Teacher writes numerals and other mathematical notation signs on the board.	• Avoid errors that are commonly made by teachers (e.g. 3 + 4 = 7 + 5 = 12)
Record work done in lesson on board or poster	Teacher records work done in class publicly for pupils to see.	• Know which work from the class is the most important to place on the record to reinforce pupil learning or to use in a future lesson
Use correct and appropriate mathematical terms	Teacher uses mathematical terms to describe various mathematical concepts. The terms are used precisely and terms that have non-mathematical meanings (e.g. face, odd) are differentiated from the mathematical meaning.	• Know terms that are used when teaching the primary school curriculum • Know what words are acceptable as synonyms and which are not. e.g. one teacher used “plan” and “pattern” as synonyms for the net of a shape. • Know which words are mathematical and which are made up e.g. is “unparallel” a mathematical term?
Elicit a mathematical term (including name of shape or number)	Use a stimulus so that pupils will use appropriate mathematical terms. These will generally be terms that the teacher believes some or most pupils already know.	• Know definitions that are mathematically accurate and understandable by pupils in the class level. • Have alternative ways to explain words that may be difficult for pupils to learn (e.g. state what the dimensions are in a 2-D shape)
Elicit the meaning of a mathematical term	Teacher uses a stimulus to prompt a pupil to explain the meaning of a word. One pupil may give the meaning or several pupils may contribute.	• Know what the term means so that the pupil‟s response can be evaluated for its accuracy and completeness
Describe or identify properties of shapes	The teacher describes the properties of a shape for the class or identifies an instance of a property of a shape. Some properties might require the teacher to give justification (e.g. a shape that is a polygon because it is closed and has straight sides).	• Know the names and properties of shapes on the primary school curriculum • Know definitions of shape properties in order to resolve disputes about properties such as the number of sides on a circle, the number of edges on a cylinder and whether or not a cone has a vertex.

Elicit properties of shapes	Teacher uses various stimuli (e.g. chart, game, open-ended or closed questions) to elicit properties of shapes. The teacher may require a pupil to give a specific property of one shape or several properties of that shape.	• Know common errors made by pupils (e.g. finding 24 corners on a cube)
Compare or differentiate between/among shapes or categories of shapes	Teacher chooses to discuss shapes in relation to one another or to discuss 2-D shapes alongside 3-D shapes. Sometimes pupils can appreciate particular properties of shapes when they are compared or contrasted with other shapes.	• Know various ways in which shapes can be compared to and contrasted with one another • Know interesting patterns in properties of shapes (e.g. Euler's polyhedral formula)
Collect data from pupils	Teacher records work done in class publicly for pupils to see.	• Know which work from the class is the most important to place on the record to reinforce pupil learning or to use in a future lesson
Use correct and appropriate mathematical terms	Teacher decides to collect data from pupils (e.g. letters in their name, favourite color) in order to represent it when teaching pupils about data collection.	• Know about different stages of data collection: posing a question, collecting and recording the data, organising the data and representing the data.
Compare or differentiate between/among different ways of representing data	Teacher discusses with pupils different ways of presenting data: e.g. bar charts, multiple bar charts, pie-charts, trend graph.	• Know different means of representing data and the benefits and limitations of each
Illustrate a property of an operation	Teacher shows pupils instances of a property such as the commutative property of addition. The teacher may or may not use the term "commutative."	• Know properties of operations and the number sets to which they apply • Know how to present the properties in ways that are comprehensible to primary school pupils
Illustrate a property of a number	Teacher shows pupils properties of numbers (e.g. odd, even, prime, square). Pupils may also be given the opportunity to test other numbers for the same property.	• Know properties of numbers that are relevant to primary school pupils • Know how to test for properties of numbers (e.g. prime number tests).
Use representations to explain operations, or other mathematical ideas	Teacher uses a representation to help pupils understand an operation (e.g. multiplication, division by fractions). The representation may be in the textbook, drawn by the teacher or by a pupil. A math sentence may be linked to the representation.	• Understand representations that are commonly used in schools to explain operations. • Match a maths sentence to the representation. • Know ways to represent equivalence of fractions, decimals and percentages
Make a mathematically accurate representation	Teacher produces a representation on the board that is accurate and that achieves its purpose of promoting understanding. This requires using appropriate resources.	• Know how to use the available resources to produce a useful representation

Teach pupils to make accurate representations	Teacher shows pupils, through instruction and/or modeling how to make accurate representations either on the board or in pupils' notebooks.	• Know what resources are available to pupils and what difficulties they have in making representations (e.g. in terms of scale or orientation)
Choose an appropriate representation for a situation	Different situations require different representations. These may vary in shape (e.g. circles or rectangles), in orientation (portrait or landscape) depending on the operation being represented, the purpose of the representation (e.g. comparison) or on the available space.	• Know what advantages different formats of representations offer and determine which one would be best for illustrating a concept.
Follow pupil explanation	Teacher listens to a pupil explaining a mathematical idea. The teacher may highlight aspects of the explanation or respond in other ways such as completing missing details.	• Know what a mathematical explanation is in general • Know what would be a good explanation in this case
Follow pupil description	Teacher listens to a pupil describe a feature of a shape or a procedure used or to be used. The description may be supported by reference to a picture or representation.	• Know the terms or other supports that can help a pupil to give a clear description which other pupils can follow
Respond to a mathematical comment, statement or conjecture from a pupil	Teacher responds to a mathematical utterance from a pupil that is related to the lesson in question or may not be. The pupil (and possibly others) will understand the point better after the response.	• Know what mathematical point lies behind the utterance • Relate the point to the pupils" mathematical knowledge
Ask other pupils to comment on a response or a statement made by one pupil	Teacher asks other pupils to respond to one pupil"s comment or answer to a question.	• Know if the initial pupil's comment or response is accurate or inaccurate
Ask a pupil to justify an answer or statement	Teacher responds to a pupil's answer to a question or problem by asking the pupil to justify the answer. Questions used may be: How do you know? Why? Why not? Are you sure? What do you think?	• Know what would serve as a mathematical justification of an answer
Ask pupil to expand on a response	Teacher asks a pupil to give a more detailed response. Typical questions might be "can you say some more about that?" or "what else springs to mind?"	• Know that a pupil's response is incomplete and what the response needs to be complete
Ask pupil to clarify a response	Teacher asks a pupil to be clearer in the response. This may happen if a pupil offers a response that is difficult to follow or contradictory. A pupil may also be asked to specify a unit of measurement.	• Recognise when an answer is unclear and know what will make it clear

Direct pupils to a mathematical definition	The teacher refers a pupil to a mathematical definition in response to a question from a pupil or to encourage independent work in mathematics.	• Know a source of definitions that are accurate and comprehensible to the pupils.
Present a mathematics task or game to pupils	Teacher presents a task to pupils making it clear to them what they are required to do and how to do it. This includes setting the conditions for the task and setting up the necessary materials. This may also include sequencing the presentation of the task so that pupils can complete one step before progressing to the next step. This task also covers the choice a teacher makes about how pupils will work on an activity. Teacher decides if pupils will work alone, in pairs, as a class-group when completing a task. Although other considerations may come into play in this task (e.g. layout of the room, attentiveness of the children etc.), part of the decision is mathematical.	• Anticipate the quantities of materials required so that all pupils can participate as required (e.g. whether or not the large cube “thousands” block is needed if using base ten materials) • Know the conditions that need to be specified to maximise mathematics learning Adjust the conditions (e.g. number and type of shapes in a feely bag) to maximise the cognitive demand of the task for pupils. • Recognise the mathematical skills that can be developed in different group formations (e.g. explaining a mathematical idea may be more likely if pupils work in groups than if they work alone). • Judge if the demands of the task are such that pupils can complete it alone or if some collaboration is needed • Know the mathematical content of the game (e.g. properties of shapes for a shape “feely bag” activity)
Draw pupils’ attention to a pattern that leads to a procedure	Teacher gives pupils various examples to complete (e.g. multiplying numbers by 10) and after pupils have completed several of them the teacher asks pupils if they have noticed a pattern.	• Know how to pick numbers that make the pattern obvious • Know what procedures can be taught in this way
Enable pupils to check if a procedure works (a) in a specific case (b) in general	Teacher asks pupils to test a procedure to determine the cases in which it works and does not work.	• Know what procedures are useful for primary pupils to learn • Know when procedures apply and when they do not • Know what cases might be particularly helpful for checking to test a procedure
Give pupils a formal algorithm to help them with calculations and explain how it works	Teacher gives pupils an algorithm that they can apply to compute operations efficiently. The teacher may be asked by pupils why the algorithm works.	• Know commonly-used algorithms for different operations (with whole numbers, integers, fractions and decimals) • Understand how and why the algorithms work

Demonstrate how to apply an informal algorithm or procedure to compute an answer	The teacher specifically demonstrates how to do a problem on the board or in a pupil's notebook. The teacher may ask questions of the pupil(s) while demonstrating the procedure	• Know what informal algorithms can be useful for particular numbers. • Know when pupils are ready for informal algorithms.
Observe and/or help publicly (e.g. on the board) a pupil apply an algorithm or procedure	Teacher requests a pupil to do an algorithm/procedure so that all pupils can see it. The teacher may observe, comment to highlight features of the procedure or help the pupil to complete it.	• Know features of algorithms that cause difficulties for pupils • Know what language to use to help pupils apply and follow the algorithm/procedure • Know what mathematical benefits can be expected to accrue to pupils from the activity
Give pupils a means to check answers	Teacher gives pupils either a criterion against which to judge their answers to questions or the teacher enables pupils to check their answers in a practical way (e.g. using a right-angle tester).	• Know range of numbers in which answers to a set of problems will fall (e.g. when dividing a whole number by a unit fraction). • Know ways of checking answers for different primary school mathematics problems (e.g. estimation, inverse operations)
Elicit or present strategies that can be used for problem solving generally	Teacher explicitly shares generic problem solving strategies with pupils that can be used for solving mathematics problems.	• Know what problem-solving strategies are helpful at primary school level • Know difficulties primary school pupils have in applying problem-solving strategies
Elicit or present methods (including alternative methods) for solving specific problems	The teacher presents specific problems to the class and discusses how they will be solved. The pupils may be asked to do the problems immediately after the discussion or at a subsequent time for independent work (as in a multi-grade class).	• Know what strategies are likely to be useful for specific topics (so that some omitted by pupils can be included). • Know how pupils respond to the topic so that the challenge in the problem is not diminished
Help pupils convert measurement quantities	Teacher assists pupils with problems that require changing from milliliters to liters; centimeters to millimeters to meters; grams to kilograms and so on.	• Know what prior knowledge pupils need to be able to convert measures • Know how to sequence instruction so that easier problems and examples precede more difficult ones
Ask pupils to estimate or predict what an answer will be	The teacher asks pupils to predict or estimate an answer before working it out in any formal way. They may be also asked if their estimate is likely to be higher or lower than the actual answer.	• Know different strategies for estimation that are used at primary school level. Know benchmarks for common measures
Ask pupils to solve a problem or to calculate mentally	The teacher asks pupils to do a problem or to calculate an answer mentally in class.	• Know how to calculate mentally • Know strategies that can be used to calculate mentally

Select suitable exercises for pupils to attempt	Teacher selects exercises related to what is being taught. Exercises may be selected from the class textbook, an alternative textbook or from another source.	• Know which exercises pupils can attempt with ease and which are likely to be more challenging • Know which exercises are likely to result in optimal pupil learning
Assign homework	Teacher assigns exercises for pupils to complete outside of school.	• Judge which exercises will reinforce what was learned in class and will be challenging enough but not too challenging for the pupils.
Modify exercises in a textbook	Teacher may supplement or omit part of the exercises in the pupils' textbook.	• Know when a change is desirable and why and how to achieve maximum benefit for the pupils' learning with the change
Devise supplementary exercise for pupils	Teacher prepares a worksheet for pupils to work on in the lesson	• Know how to prepare the activity so that all pupils will learn some mathematics and achieve success
Provide work for pupils who finish early	Teacher assigns work to pupils who successfully finish class work while other pupils are still completing the class work.	• Know what would be a suitable extension of the main topic in the lesson
Indicate to a pupil that an answer is correct	Teacher evaluates an answer to a question or to a problem and tells the pupil that the answer is correct. This may be accompanied by a compliment or an instruction to keep going.	• Know or work out the answer to the question or problem
Indicate to a pupil that an answer is incorrect	Teacher evaluates an answer to a question or to a problem, and tells the pupil that the answer is incorrect and does not follow-up the answer in any mathematical way.	• Know or work out the answer to the question or problem
Tell or show pupils the answer to a question	Teacher tells pupils the answer to a question or shows it to them in the form of a picture, diagram or object.	• Know how to get the answer • Know the importance of the answer in relation to the solution of the particular problem (Lampert, 1990)
Share one pupil's (or one group's) work with the rest of the class	Teacher directs all pupils' attention to work done by one pupil. This may be because the pupil has used a novel approach or because the pupil has done particularly good work. Alternatively it may be because the pupil has made a common error which other pupils should avoid making	• Recognise what constitutes mathematical work that could benefit other pupils' learning through sharing.
Check if a pupil understands	Teacher checks if an individual pupil understands a term or concept or a procedure that arises in class by giving a task or asking a question (other than "Do you understand?"). The term or concept or procedure may be one that arose informally in the lesson.	• Know what constitutes understanding of the term or concept • Know what would be a suitable task to assess understanding

Demonstrate how to investigate properties of a shape	Teacher shows pupils ways to investigate the properties of shapes. These may include using equipment (e.g. right-angle testers, or a ruler) or using a system (e.g. to count the number of edges on a rectangular prism).	• Know ways of investigating properties of shapes that are appropriate for use with primary school pupils.
Use class materials or activities to model a mathematical concept	The teacher uses activities or materials to explicitly model a mathematical concept.	• Know which materials and activities are best for teaching specific concepts • Know how to link the concept and the materials
State the purpose and use of mathematics education equipment	Teacher knows a wide range of materials that are available for teaching mathematics. These include materials that are available in the school. They also include materials that are not in the school because a pupil may see reference to them in a textbook and ask what they are.	• Know a wide range of materials that are available to support teaching mathematics in primary school • Know the strengths and limitations of different materials for use in teaching various topics
Identify appropriate equipment for doing a mathematical task	Teacher knows materials that pupils use when doing primary school mathematics. This includes items such as compasses and protractors.	• Know what materials can be used for particular mathematics tasks in primary school.
Introduce materials or visual aids to the pupils	The teacher gives pupils an overview of the mathematics materials, pointing out the key features. There may also be time for pupils to freely explore the materials.	• Know the key features of the materials about which pupils need to know
Ask pupils to use materials in a specific way for a specific purpose	The teacher gives an instruction to pupils to perform a specific activity using the materials, directed towards learning an aspect of a mathematical topic.	• Know how materials can be used to help teach mathematics concepts
Explain inadequacies in materials or drawings being used	The teacher may not always have the ideal equipment for the task in hand and, therefore, may need to explain to pupils why and the way in which the materials are inadequate.	• Assess the shortcomings of the available materials • Judge whether the inadequate materials are preferable to using no materials
Draw shapes on the board or on a poster	Teacher uses available resources (e.g. rulers and markers or interactive whiteboard) to produce clear shapes for class discussion.	• Know how to use the available resources to produce the shapes • Avoid choosing stereotypical shapes, e.g. an equilateral triangle, unless it is specifically required
Use materials or a picture to confirm, question or understand a pupil response	Teacher uses materials or a picture with the pupils to either confirm the pupil's answer, to question it or to understand why or how the pupil came up with the answer.	• Know how to connect a written or oral answer with concrete materials

Tasks done by teachers that would not be observed in videotapes of lessons		
Sample Dimensions of Work of Teaching	**Description of Task**	**Sample MKT Demands**
Teach number facts to pupils	Children need to remember basic calculations in all four operations (within twenty for addition and subtraction and within one hundred for multiplication and division)	• Know number and operation properties (e.g. commutative property; additive identity property) that make learning the number facts easier.
Correct (mark) pupils' homework or class work in mathematics	The teacher collects pupils' mathematics copies and marks them after school or at home. Pupils are given feedback on their work and the teacher can evaluate how well pupils have grasped a mathematical idea.	• Know the answers to the problems assigned Work out pupils' strategies or errors when pupils are not present • Identify common patterns of errors • Know what feedback will be helpful to pupils
Contribute to writing a school plan in mathematics	Schools are required to have a written plan stating how each subject is taught throughout the school and although they are prepared at school level, the aspiration is that all teachers contribute to the plans.	• Know how each strand (number, algebra, shape and space, measures and data) of the mathematics curriculum develop throughout pupils' years of primary school
Contribute to staff discussions about mathematics	Discussions about mathematics can be informal or formal and the topics can be wide-ranging from figuring out a solution to recommending materials for teaching a particular topic to a problem to the language that is used when teaching subtraction.	• Know how to solve primary school mathematics problems • Know about materials that are suitable for teaching various topics • Know language to talk about teaching mathematics
Recommend a textbook to be adopted by the school	Select a suitable mathematics textbook for use in the school. Typically about three options are available at a given time and once chosen textbooks can be used in a school for several years.	• Know how to source and use frameworks for evaluating textbooks (e.g. for sequence and presentation of topics, cognitive demands of tasks and so on).
Write long-term and short-term plans for teaching mathematics	Plan the material that will be taught during the specified time period. This will include topics to be taught, exercises to be used and plans for assessment	• Source and choose problems that will help pupils learn the planned content
Keep a record of mathematics taught to pupils	Record the material that has been taught in a specified period of time.	• Know how to document mathematical learning in a way that is useful to colleagues
Purchase and make manipulative materials and visual aids	Choose materials that will help to teach particular topics.	• Know what features of materials are mathematically sound and will help pupils acquire the desired concepts

Design tests to assess pupils' progress	Many teachers give mathematics tests regularly throughout the school year and especially at periods such as the end of term. Many of these are designed by the teacher.	• Know how to write items that will test what pupils have learned and through which all pupils can achieve some success
Administer a standardised mathematics test to pupils	Many schools administer standardised mathematics tests to their pupils once a year. This is used to monitor learning and to identify pupils who may need additional support in learning mathematics.	• Know how to interpret the result of the test for each pupil in light of their work in mathematics during the year
Document a child's progress in mathematics in a school report or discuss progress at an individual parent-teacher meeting	School reports vary but most reports require at least a box to be ticked summarising a pupil's progress in mathematics. A meeting with a pupi's parent(s) requires more detail about mathematics learning.	• Know how to summarise concisely what a pupil has learned in mathematics during the year. • Know strategies to recommend for the pupil to make further progress in mathematics
Answer parents' questions about mathematics teaching	Many teachers meet parents at the start of the school year to outline their expectations and plans for teaching all subjects (including mathematics) during the year. This is an opportunity for parents to ask questions about approaches that will be used.	• Explain teaching approaches and the rationale for them in language that parents understand.

Appendix 2

Number of teachers in each stratum chosen for the sample.

Stratum	Dublin	Leinster (ex. Dublin)	Munster	Connacht/ Ulster	Total
Breaking the Cycle (Urban)	1	0	0	1	2
Breaking the Cycle (Rural)	0	0	0	1	1
Disadvantaged	3	2	2	1	8
Gaeltacht School	0	1	1	3	5
Gaelscoil	1	1	2	1	5
None of the above categories	7	19	22	18	66
Total	12	23	27	25	87

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