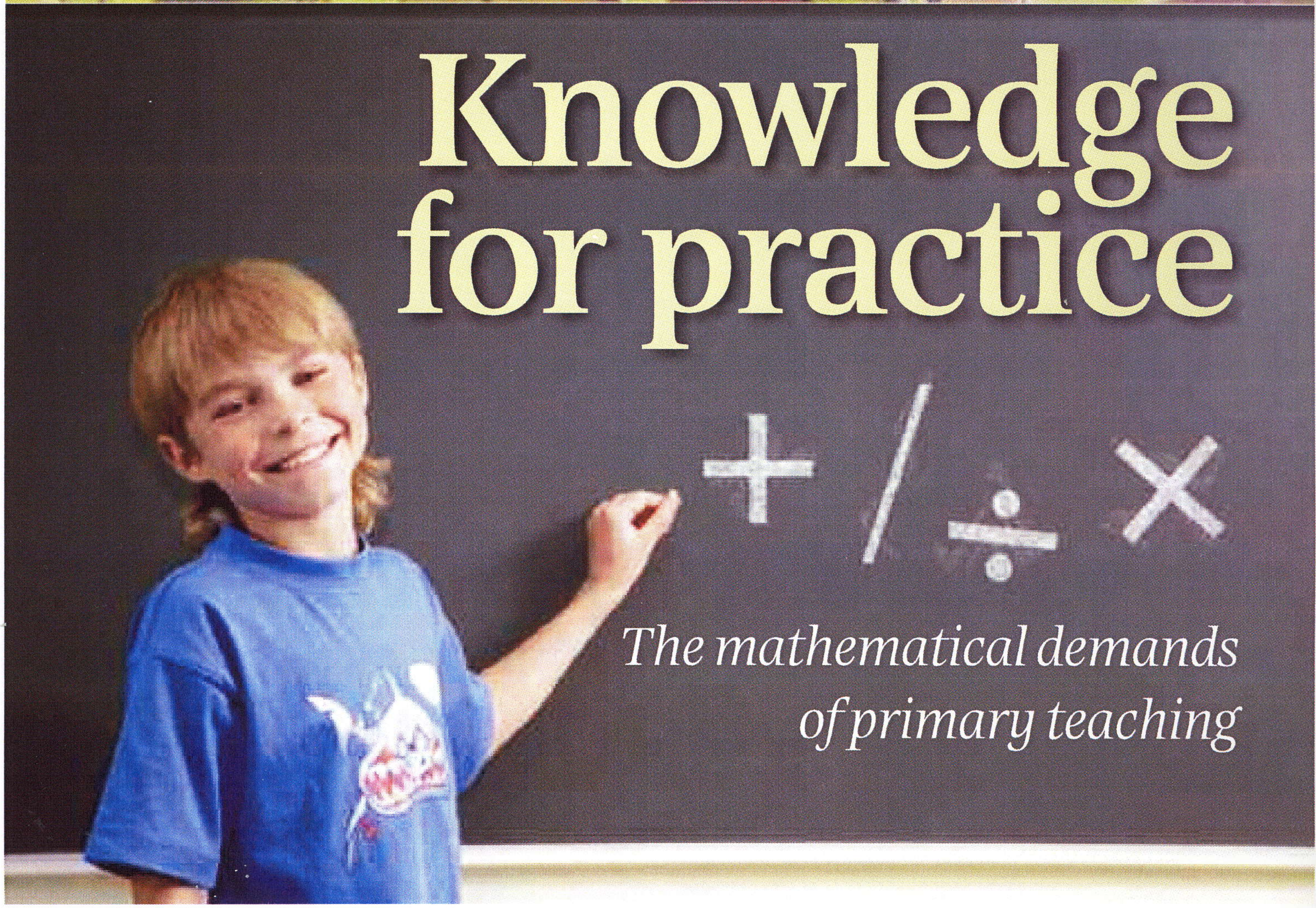


Knowledge for practice

The mathematical demands of primary teaching

In a recent maths lesson, one of Fiona's pupils raised a hand and said: "Remember yesterday when we were talking about parallelograms and you drew one on the board? Well, my mam says that a rectangle is a parallelogram, and so is a square." With the other pupils listening, Fiona was unsure how to respond. She had drawn and described a parallelogram based on the definition in the maths textbook, which said that "a parallelogram is a rectangle that is pushed out of shape." That was how she had always introduced parallelograms. But reflecting on her pupil's comment, it made sense to Fiona that rectangles themselves and squares might be parallelograms too, because opposite sides of the shapes are parallel. She told the class that it was an interesting idea, one they would return to the next day.

That evening Fiona surfed the internet and learned that a parallelogram can be defined as "a quadrilateral with opposite sides parallel (and therefore opposite angles equal)."[1] So squares, rectangles and even rhombuses are parallelograms. The pupil was right. Fiona felt annoyed that the definition in the maths book was incomplete. She would like to have known the full definition when teaching the lesson.

> **The pupil was right. Fiona felt annoyed that the definition in the maths book was incomplete**

In a study of 40 videotaped maths lessons taught by ten Irish teachers, I was struck by how much mathematical knowledge teachers use in their work. Six incidents from the lessons illustrate the kind of mathematical knowledge teachers use when teaching. After outlining each example, I analyse the mathematical knowledge that underpinned the teacher's work.

Clíona – Eliciting different ways to solve a maths problem

Teaching instance: The first teacher I introduce is Clíona, who was working with pupils to solve this problem.

How much would ¼ kg of mushrooms cost if the price was €0.62 per 100g?

Three pupils suggested different ways to calculate the cost:

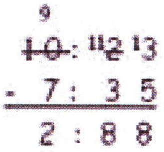
1. Multiply €0.62 by two and add half of €0.62.
2. First find the cost of 25g by dividing €0.62 by four, and then multiply the answer by ten.
3. Find what a kilo of mushrooms costs by multiplying €0.62 by ten, and divide the answer by four.

Mathematical knowledge: First, Clíona had to know that finding the cost of 100g, 100g and 50g – which add up to 250g or ¼ kg – would give the total cost. For the second suggestion she needed to know that one quarter of 100g is 25g and that 25g is one tenth of 250g. To follow the third method Clíona had to know that 100g is 1/10 kg and that when you know the cost of 1kg, you can calculate the cost of ¼ kg.

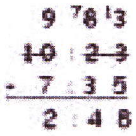
Implication: Clíona probably prefers to calculate the answer using one method; but as a teacher she needs to follow different approaches, determine if pupils understand them, and check if they will yield the right answer.

Eileen – Anticipating difficulties pupils will have

Teaching instance: In another lesson Eileen's pupils were using a train timetable to find the duration of various journeys. One train leaves Dublin at 07:35 and arrives in Cork at 10:23. Eileen anticipated that some pupils would incorrectly calculate the journey time as follows:



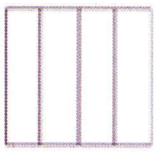
Mathematical knowledge: Rules for subtracting tens and units were used to calculate the answer; but because there are only 60 – and not 100 – minutes in an hour, the calculation is wrong. Eileen knew that the right answer would be found by exchanging one hour for 60 minutes and adding them to the 23 minutes:



Implication: Eileen combined her knowledge of mathematics and her knowledge of pupils to pre-empt this error: "Watch when you're doing your regrouping. 60 minutes is not like the hundreds, tens. So, you have to watch out for that."

Brendan – Representing mathematical ideas

Teaching instance: Brendan, asked a sixth class pupil to draw a diagram to represent the division problem 1 ÷ ¼ [2]. The pupil drew the diagram below and hesitated, unsure of which panels to shade.



Following a brief class discussion Brendan asked the pupil if he was happy with the drawing. He responded:

"Yeah, it's just the answer is all of them, not just one. It's usually one. Because if you're quartering it, the answer is one of them, but if you're dividing by a quarter it's all of them."

Mathematical knowledge: Depending on how it's shaded, the diagram can be used to represent two calculations that are easily confused: a quarter of one, and one divided by a quarter. These are written numerically as 1 ÷ 4 (or ¼ of 1) and 1 ÷ ¼ respectively. The answer to the first one is ¼ and the answer to the second is 4.

Using different diagrams to help pupils understand mathematical concepts is part of the "mathematical work of teaching". Representing such concepts makes demands on a teacher's mathematical knowledge

Brendan needed to understand the distinction to respond to the pupil's uncertainty. If one panel is shaded, it can represent the calculation 1 ÷ 4, and if all four panels are shaded, the diagram can represent the calculation 1 ÷ ¼.

Implication: Using different diagrams to help pupils understand mathematical concepts is part of the "mathematical work of teaching" (Ball and Bass, 2003). Representing such concepts makes demands on a teacher's mathematical knowledge.

Lorraine – Evaluating mathematical examples

Teaching instance: Lorraine was teaching about priority of operations. The rules are often summed up by the mnemonic BOMDAS, indicating that calculations within brackets precede "of" calculations (eg ¼ of 8), which precede multiplication and division, which precede addition and subtraction.One problem in the book was 37 – 56 + 28. Pupils found it difficult to subtract 56 from 37.

Mathematical knowledge: In order to help her pupils, Lorraine needs to know that 37 – 56 + 28 is the same as 37 + (–56) + 28. This means that the pupils can use the "any-order" property [3] to first add 37 and 28, and then add –56 (or subtract 56) to give the correct answer of 9. This textbook exercise is problematic in at least three ways:

- Because it uses only addition and subtraction – which have equal priority of operation, and you operate on the numbers from left to right – it's a poor example on which to practise applying priority of operation rules.
- Given how the calculation is written, pupils might incorrectly interpret the calculation to be 37 – (56 + 28), which would give the wrong answer of –47.
- For pupils who are not confident adding negative numbers, the logical way to do this problem is (37 + 28) – 56; but this means that they must first be able to apply the "any-order" property in order to solve the problem.

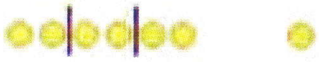
Implication: Many of us have come across similar exercises in textbooks, realising too late that the exercise is unsuitable. Recognising poor – and good – mathematical examples to illustrate and practise new ideas requires mathematical knowledge.

Linda – Applying a mathematical definition that is accurate and understandable

Teaching instance: Linda teaches senior infants. In a lesson about the number seven she reminded pupils of the class's definition of odd number: "When we were sharing out the teddies, we couldn't, no matter how we tried, we couldn't share them out so that the two boys both had the same." Pictorially this could be represented as follows:

Mathematical knowledge: Mathematically the definition would be expressed as 2k + 1, where k is any whole number (ie 0, 1, 2, 3, 4, ...). Linda's expertise was in selecting a mathematically precise definition and adapting it so that senior infants could understand it. If a pupil asked whether 0 is odd or even, the definition could be used to respond: 0 is a whole number; if 0 plastic teddies are shared equally between two children, no teddy is left over so 0 is not odd; and because all whole numbers are either odd or even, 0 must be even. Because Linda chose objects that cannot be broken in half, pupils could not suggest halving one teddy to share them equally.

Another definition for odd numbers would be a quantity that cannot be fully partitioned into groups of two (Ball and Bass, 2000). Mathematically, this is written as k2 +1, where k is a whole number: when the maximum number of possible pairs is made, one is still left over. This would be illustrated as follows.



Sometimes pupils learn that an odd number is a number with 1, 3, 5, 7, or 9 in the units place; but based on that definition, the number 17.5 could reasonably be considered to be odd (which it is not). The number system needs to be confined to whole numbers initially, and to integers for older pupils.

Implication: Teachers need to use mathematical definitions that are both comprehensible to their pupils and mathematically accurate.

Veronica – Analysing shapes in the environment

Teaching instance: In another classroom, Veronica asked her pupils to name examples of cylinders. One pupil looked at the music table and suggested that castanets and bongo drums are cylinders. Veronica was unsure whether to accept or reject the suggestion.

> **... teachers have typically had few opportunities to acquire the knowledge that can be used as a resource in the work of teaching maths**

Mathematical knowledge: What mathematics might Veronica need to know to guide her response? Presumably the pupil was thinking that each castanet consisted of two thin cylinders tied to each other by leather or string. If castanets are held horizontally, the outer face of the upper castanet is usually not a complete circle, ruling it out as a cylinder. Even if it were circular, the outer part is usually convex and the inner part is usually concave, meaning that the shape is not bounded by two parallel planes, which eliminates it from the set of cylinders.

Any bongo drums I've seen are conjoint pairs. Even if one drum is considered, however, it does have two parallel planes but the diameter of the upper circular plane is wider than the diameter of the lower circular plane. Therefore, the bongo drum resembles more the shape of a truncated cone than a cylinder.

Implication: Several reasons existed for Veronica to exclude both instruments as examples of cylinders. But analysing shapes in this way, in the middle of a mathematics lesson, is no trivial matter.

Teachers' specialised knowledge of mathematics

What happened in each exchange described above hinged on the teacher's knowledge, not of teaching methods, but of mathematics. What was needed was a kind of applied mathematical knowledge that is held by teachers. Like teachers, engineers, nurses, plumbers and computer programmers all use mathematical knowledge. In each case the mathematical work done determines the knowledge that is needed.

My study of Irish teachers was inspired and informed by the work of Deborah Ball and her colleagues at the University of Michigan who label the type of knowledge that teachers need as "mathematical knowledge for teaching" (Ball and Bass, 2003; Ball, Thames, and Phelps, in press). Some examples will illustrate how Ball and her colleagues think of this knowledge.

Specialised knowledge of mathematics used when teaching subtraction

Imagine a teacher working on subtracting with renaming, using the numbers 82 – 35.

- The teacher needs to know that the answer is 47, but this mathematical knowledge is held in common with people in other jobs. A teacher must know more.
- A teacher needs to be able to follow different algorithms[4] used by pupils to calculate the answer, such as decomposition (regrouping) and equal additions ("borrow and pay back"). Familiarity with different algorithms is needed if a pupil joins the class from another school, or if parents show pupils a different method. Only teachers need to be able to follow multiple algorithms for calculating and it is an example of specialised content knowledge.
- A teacher needs to know why a pupil might produce the incorrect answer 53 to this problem – a combination of knowing mathematics and knowing pupils.
- Finally, a teacher needs to know the relative merits of using visual aids such as a hundred square, a number line, or base ten materials and a notation board to represent the problem. This combines knowledge of mathematics and knowledge of teaching.

Acquiring professional knowledge of mathematics

Although it hardly comes as a surprise to teachers that they use mathematical knowledge when they teach mathematics, relatively little was known until recently about the specialised nature of this knowledge. Consequently, teachers have typically had few opportunities to acquire the knowledge that can be used as a resource in the work of teaching maths. Many teachers have had few opportunities to study mathematics content after completing their Leaving Certificate. Secondary school maths cannot be expected to adequately prepare teachers for the mathematical work they do – work that includes following how other people solve problems, knowing typical pupil misconceptions, representing ideas in multiple ways, and analysing environmental shapes mathematically. These tasks of teaching are mathematically demanding, but learning more about teachers' professional knowledge of mathematics can help to strengthen the profession of teaching.

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1. http://mathworld.wolfram.com/Parallelogram.html
2. This problem might arise in the context of finding how many "quarter-pounders" can be made from a 1 lb mixture of meat; or how many quarter-hour appointments a doctor can fit in one hour.
3. The "any-order" property is a combination of the commutative and associative properties of addition and in this case it works as follows: $37 + (-56) + 28 = 37 + ((-56) + 28) = 37 + (28 + (-56)) = (37 + 28) + (-56)$
4. Algorithms are methods for calculating, such as repeated subtraction or equal sharing in division.