Engaging *All* Students in Challenging Mathematical Work:

Working at the Intersection of Cognitively Challenging Tasks and Differentiation

During Lesson Planning and Enactment

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Abstract

Drawing upon and extending work undertaken by Silver and colleagues, in this chapter we bring together two fields which have largely developed on parallel tracks: engaging students in mathematically challenging tasks and differentiating teaching to meet all students' needs and readiness levels. By working at the intersection of these two lines of research, we attempt to understand the implications of this dual focus for teachers' practice. We do so by discussing the entailments of working at the nexus of challenging tasks and differentiation in mathematics lessons during lesson planning and enactment (i.e., task launching, student autonomous work, and whole-class discussion). Unpacking teaching in this way offers insights to researchers and contributes to supporting teachers in addressing both excellence and equity in their teaching, two significant educational aspirations in several countries around the world.

Keywords: challenging tasks, cognitive activation, cognitive demand, differentiation, mathematics, teaching entailments, unpacking teaching.

Introduction

Engaging students in cognitively challenging work in order to foster their mathematical learning and reasoning has been systematically emphasized and empirically corroborated in both older (Doyle, 1983; Henningsen & Stein, 1997; Stein & Lane, 1996) and more recent studies (Baumert et al., 2010; Boston & Smith, 2011; Kunter et al., 2013; OECD, 2020). Despite its importance, cognitively challenging teaching is rarely documented in contemporary mathematics classes. For example, in an examination of algebra teaching across eight countries, the *Global Teaching InSights* study (OECD, 2020) reported that students only occasionally engaged in challenging work. On a scale from 1 (lowest score) to 4 (highest score), the mean score of cognitively demanding teaching observed ranged from 1.36 in Chile to 2.52 in Japan—a finding that aligns with earlier international comparative studies (e.g., see Hiebert et al., 2003; Stigler & Hiebert, 1999). Together, these results suggest that such teaching is either rarely prioritized or difficult to enact.

Yet, in a pioneering project studying the cognitive demands of tasks in mathematics classrooms, Silver and colleagues attempted to "revolutionize the possible," by engaging students in urban disadvantaged areas in cognitively challenging work (see Silver & Stein, 1996). Convinced that this type of teaching would benefit not only students from affluent areas, but also students in economically disadvantaged communities, Silver and colleagues launched *QUASAR* (Quantitative Understanding: Amplifying Student Achievement and Reasoning), an ambitious project aiming to offer students from disadvantaged backgrounds opportunities for high-quality, cognitively demanding, mathematics teaching. Against all odds, data from this project (Silver & Stein, 1996; Stein & Lane, 1996; Stein et al., 2007) confirmed that such teaching is indeed feasible and beneficial for student learning. In particular, positive outcomes were found on several metrics, including (a) performance on a challenging test of reasoning, problem solving, and mathematical communication, (b) performance on a typical standardized test (NAEP), and (c) students' subsequent enrollment in algebra classes. As such, the results of the QUASAR project challenged "a pernicious belief that high-level mathematical objectives and performance expectations are not appropriate for all students," especially for students from low-income households or students who face difficulties in learning mathematics (Silver & Stein, 1996, p. 479).

Assisting *all* students to learn important mathematical ideas and gain proficiency with mathematical concepts and processes has been a persistent concern for Silver and colleagues. This is suggested, among other things, by Silver and Kenney's (2016) *Useful Research on Teaching Important Mathematics to All Students*, an edited volume of 24 articles aiming to communicate to practitioners in simple language important research findings. Doing so was thought to represent "an important strategy for increasing the quality of education" (p. v).

Despite Silver and colleagues' pioneering work in this area, the lack of documented evidence of promoting mathematically challenging work for all students suggests that further explication is required of what this type of teaching entails in practice. This becomes particularly important, taking into consideration the findings reported above and the increasing diversity of student populations in contemporary classes, which imply that enacting such teaching for all students requires mathematics educators to carefully unpack and map its essence in order to help teachers navigate this complex terrain. It is also critical, given the dual emphases on excellence and equity in educational systems worldwide (see Kyriakides et al., 2018; OECD, 2016).

Toward this end, we bring together work undertaken in two areas that are more frequently studied independently than together: work on mathematically challenging tasks (see Kunter et al., 2013; Stein et al., 2007; Sullivan et al., 2015) and work on differentiation (Tomlinson, 2014). Bringing together these two research strands as two overlapping viewing lenses can reveal what is entailed in engaging all students in cognitively challenging work during both lesson planning and lesson enactment. As such, this work can have important implications for both research and practice.

Research-wise, it responds to recent calls (cf. Hiebert & Stigler, in press) to consider *systems* of teaching dimensions instead of treating them separately; it also resonates with theoretical models that consider differentiation and adaptation to be interwoven with other dimensions [e.g., in Charalambous and Praetorius' (2020) *MAIN-Teach* model, differentiation and adaptation are thought to be interwoven with the dimension of cognitive activation, with the latter corresponding to working on challenging mathematics tasks]. This represents a move in the opposite direction compared to that undertaken over the past decades: for years, teaching dimensions have been treated as the strands of a rope and scholarly work has focused on decomposing them in order to better understand them (see Figure 1). Without underestimating the benefits accrued from this approach, we argue that it is time to consider the strands as intertwined and focus on their interactions instead of treating them as separate or independent components. Doing so might illuminate the complexity inherent in teaching (cf. Cohen, 2011); it could also help explain its effects on student learning, which, as past research has suggested, are lower than expected (cf. Lindorff et al., 2020; Muijs et al., 2014; Scheerens, 2016). In this respect, this chapter is likely to be of interest to scholars working on understanding and studying teaching quality and its effects on student learning.

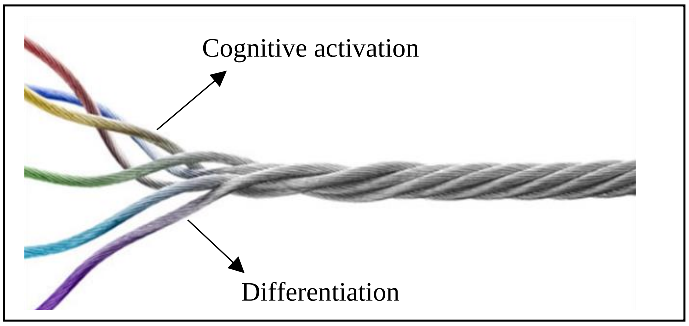


Figure 1. Teaching as a rope consisting of strands representing different teaching dimensions indicating the two that are the focus of this chapter.

Practice-wise, this chapter contributes an illustrated example of how mathematical challenge and differentiation can be integrated through various parts of a lesson. Such teaching requires deliberative work from planning the lesson to the execution of each lesson phase. Hence, the teacher must maintain a dedicated, persistent, and simultaneous focus on differentiation and challenge throughout the lesson. Documenting the entailments of this work can inform teacher preparation programs and professional learning programs for in-service teachers, especially given that this type of teaching—often identified as ambitious mathematics teaching (Cohen, 2011; Lampert et al., 2010)—is expected from both experienced and novice teachers (Deunk et al., 2015). Therefore, this chapter offers insights to mathematics teacher educators, teacher professional developers in mathematics, and teachers of mathematics.

Engaging All Students in Challenging Work: Insights from Prior Studies

In this section, we first define key terms pertaining to challenging mathematics work and differentiation. We then survey prior studies that have worked at the intersection of these strands, and identify research gaps and open issues.

Defining Key Terms

Different terms are used to capture challenging work. These include cognitively demanding tasks as used by the QUASAR scholars (e.g., Stein et al., 2000), challenging tasks (e.g., Sullivan et al., 2012) and work that promotes students' cognitive activation (e.g., Baumert et al., 2010; Kunter et al., 2013). Although not identical, we use these terms interchangeably to denote work around mathematical tasks that engage students in high-level thinking and mathematical reasoning. Such understanding includes processing multiple pieces of information and making connections among them, choosing which strategies to apply, explaining the strategies selected, constructing mathematical arguments, responding to others' arguments, and justifying one's thinking to the teacher and classmates (NCTM, 2000; Stein et al., 2000; Sullivan et al., 2012).

Differentiation is based on the premise that students achieve their full potential when teachers plan and enact lessons that accommodate differences among students (Corno, 2008; Tomlinson, 2014).[1] This requires matching learning targets, tasks, activities, resources, and learning support to learners' needs, readiness levels, learning profiles, and rates of learning (Beltramo, 2017; Stradling & Saunders, 1993). Teachers can differentiate four different classroom elements: the content (what is being taught), the process (how it is being taught), the product (evidence that students are learning), and students' learning environment (how the classroom works and feels) (Tomlinson, 1999).[2] Unlike individualized instruction (cf. Slavin et al., 1984), where teachers develop individual tasks or lesson plans for every student in the classroom, in differentiated instruction the teacher requires (groups of) students, and/or the whole class, to work with key ideas at varied levels of complexity and with varied support systems (Tomlinson, 2008). As such, this conceptualization alludes to the importance of considering the confluence of cognitive activation and differentiation, to which we turn next.

Working at the Nexus of Cognitive Activation and Differentiation

During recent decades, research on cognitive activation and differentiation seems to have largely developed on parallel tracks. Research focusing on cognitive activation has highlighted the importance of engaging students in cognitively challenging work; however, missing from such research was the variation in students' level of readiness, which pervades a typical classroom, and how this variation may interact with the stated objective of promoting cognitive engagement (see a similar argument in Tekkumru-Kisa et al., 2020).

[1] We purposefully use the term "differentiation", instead of the term "adaptation", since the latter term represents a wider notion that includes approaches beyond differentiated instruction considered herein, such as individualized instruction, personalized learning and open instruction (cf. Dumont, 2018, p. 54).
[2] Teachers may not differentiate all four components within a single lesson.

Research on differentiation has focused primarily on elaborating theoretically the importance of differentiation, forms it takes, and how they can be enacted in the classroom (cf. Delaney, 2016; Tomlinson, 2014). The crucial question of how teaching can be differentiated to engage all students in cognitively challenging tasks at their own level has not received explicit, consistent attention. Despite the non-intersecting paths that these two strands of research have tended to follow, the importance of working at their intersection has been emphasized by a small body of studies; most of them either theoretically supported or empirically validated the critical role of modifying task demands to meet different students' needs as a key element of reinforcing student learning.

In an explicit early reference to the importance of modifying task demands for different students, Good and Power (1976) used the term "high task differentiation" to identify one of three general conditions needed to maximize student achievement. Challenging the tendency to consider the class as a single unit and to search for the effects of certain teaching aspects on student learning, these authors advocated manipulating the difficulty level and other aspects of the assigned task demands for different groups of students. Toward this end, they proposed different strategies including posing questions that vary in cognitive demand and making different materials available in the classroom. Subsequent studies empirically corroborated the importance of modifying task complexity for student learning, showing its positive effects for either advanced students when cognitive complexity is aggravated (e.g., Diezmann & Watters, 2000) or for struggling students when cognitive complexity is reduced (Boaler, 2008; Stein & Lane, 1996).

Several scholars have implicitly underlined the importance of combining cognitive activation and differentiation. Corno (2008), for example, emphasized ongoing monitoring of students to ascertain the support they need, especially when working on challenging content. Similarly, Patrick and colleagues (2012) noted that supporting students' different needs has meaning when students engage with the learning content in depth and are cognitively stimulated, whereas Dumont (2019) claimed that differentiated (or more broadly adaptive) instruction needs to challenge every student's thinking in order to be effective for student learning. The notion of cognitive support recently introduced by Kleickmann and colleagues (2020) could also be thought to lie at the intersection of cognitive activation and differentiation, since it refers to adjusting the support provided to students when the latter are involved in challenging work to support students' progress and learning.

Perhaps the most systematic work on explicitly attending to both notions and offering particular tools for doing so has been conducted by Sullivan and colleagues in Australia.[3] Coining the term "enabling and extending prompts" (or enablers and extenders), Sullivan and colleagues (2006) proposed that teachers can support students to access, understand, and address mathematically challenging tasks and productively extend their thinking by modifying aspects of task complexity: enablers tailor the challenge to support the learning of students who experience difficulties accessing the assigned task, whereas extenders further extend the challenge and thinking of students who have completed the task (Sullivan et al., 2016). A series of studies conducted by Sullivan and colleagues in primary grades (Sullivan & Davidson, 2014, Sullivan et al., 2009, 2016), secondary grades (Sullivan et al., 2012; Sullivan & Mornane, 2014) and in both grade levels (Sullivan et al., 2015) highlighted the potential of this approach for supporting teachers' attempts to engage different groups of students in cognitively challenging work; it has also documented positive changes in students' performance and stance toward complex mathematical work.

In sum, only a small body of prior studies has explicitly combined considerations of cognitive activation and differentiation. Yet, such studies have typically either focused on a

[3] Other scholars (e.g., Little et al., 2009) have also worked on exploring how complexity can be adapted for different groups of students. However, their work was more limited in scope than that of Sullivan and colleagues.

limited aspect of teachers' work at the intersection of these two fronts—as the research on enablers or extenders suggests—or have not systematically examined what it means to concurrently attempt to address both goals. In this study, we probe the boundaries between these two strands, by working at their intersection in order to contribute towards their integration—a need that has recently been highlighted by Walter Doyle, one of the first scholars who systematically discussed the role of challenging tasks in supporting student learning (Doyle, 1983, 1988). Working with colleagues, Doyle has emphasized the need to study how teachers select, adapt, and enact challenging tasks in ways that make these tasks accessible to *all* students, without removing the challenge (Tekkumru-Kisa et al., 2020). Reviewing almost 40 years of research on challenging tasks, these scholars have identified this need as a key open issue in the field. Considerations of task complexity have been identified in research on differentiation, as the previous section suggested—yet, without any systematic and explicit attempt made to integrate both lines of research. Hence, we argue that exploring how cognitive activation and differentiation are interwoven can help us better understand how challenging tasks and differentiation can function in mutually beneficial ways, since, as we maintain, true differentiation cannot exist without challenging students appropriately and cognitive activation cannot fully meet its potential unless *all* students are challenged at an appropriate level.

Apart from its theoretical contribution, this work has implications for practice, as suggested by studies documenting teachers' difficulties when attempting to engage with even a single aspect of this challenging work, such as using enablers and extenders (e.g., Charalambous et al., 2022a; Hodgson, 2019; Minas, 2019; Sullivan et al., 2015) or scaffolding low achievers when working on challenging tasks (e.g., Pfister et al., 2015). Given the narrow focus to date of studies on engaging students in cognitively challenging work and/or on differentiation, it is timely to systematically unpack the entailments of interweaving cognitive activation and differentiation in practice. Carefully decomposing the entailments of this work can help better understand the intricacies of such work, and in so doing, better support teacher educators' attempts to help teachers implement this type of teaching.

Aim

This chapter uses classroom episodes (supported by records of practice, Lampert & Ball, 1998) to theoretically elaborate and exemplify what is entailed in working on challenging tasks with *all* students during lesson planning and enactment. Our work sets out from the premise that cognitive challenge and differentiation operate in tandem. Yet, as argued above, our understanding (as a community) of how challenge and differentiation play out in situations designed to address them together is, at best, incomplete. Hence, embarking on the theoretical elaboration of their interplay and outlining the entailments for the work of teaching to appropriately respond to this dual challenge is worthwhile. In this chapter we ask, *How does working at the intersection of challenging tasks and differentiation in mathematics lessons look during lesson planning and lesson enactment?* In the next section, we identify and explain decisions that guided our exploration of this question.

Decisions for Studying Cognitive Activation and Differentiation in Practice

We made three decisions that were crucial to the chapter's structure. First, data used in the chapter are classroom episodes loosely based around a week-long summer school for fifth-grade students taught by Mr. Shea (pseudonym), an experienced elementary school teacher and teacher educator. The summer school was designed to illustrate teaching for cognitive activation and differentiation. Second, the key unit of analysis was a lesson, and we parsed the lesson into four phases: lesson planning, launching a task, student autonomous work, and whole class discussion. Third, within each lesson phase we identified teacher practices and moves, accompanied by a rationale that explained how they contributed towards the goals of cognitive activation and differentiation. We now elaborate on each decision.

Classroom Episodes

The 25 fifth-grade students in the dedicated mathematics summer school came from nine different schools in Dublin, Ireland. The vast majority of primary-school students in Ireland are taught by teachers who use textbooks daily and for whom textbooks are the main planning resource (Eivers et al, 2010). Such textbooks typically contain few tasks with high cognitive demand (Charalambous et al, 2010), and consequently, it is likely that few students had extensive experience working on the kind of tasks used in the summer school. The school was modeled on Deborah Loewenberg Ball's elementary mathematics laboratory (Shaughnessy et al., 2017) and was observed by mathematics educators from four countries.[4] The teacher teaching in the summer school has over thirty years' experience as a teacher and a teacher educator. His approach to teaching is one where students are encouraged to take responsibility for their personal learning and for sharing their learning with peers through listening, questioning, responding and explaining their ideas, confusions and solutions.

Extensive records of practice in the form of video, student work, and teacher interviews were collected and utilized to develop the classroom episodes used in this chapter. Drawing on Blanton and Kaput (2005), in this chapter we define classroom episodes to be units of conversation in which a practice or move occurred that promoted cognitive activation and differentiation.

Unit of Analysis and Lesson Parsing

When investigating innovative teaching, one can study various units or durations of teaching, such as a year (e.g. Lampert, 2001), a lesson (e.g. Stigler & Hiebert, 1999) or a

[4] This summer school was part of project EDUCATE (see https://ucyweb.ucy.ac.cy/educate/en), a European project with an explicit focus on cognitive activation and differentiation.

critical incident (e.g. Griffin, 2003). We chose a lesson as the key unit of analysis given its widespread use in the organization of school teaching and its potential for teacher learning, frequently through lesson study (Stigler & Hiebert, 1999).

Many models of lesson structure have been identified (Maulana et al., 2012). Ideas of lesson structure are traced back to Herbart and his follower Rein, who identified five steps in a lesson: preparation, presentation, association, generalization and application (Ivie, 2007). Although in practice lessons may look very different across settings, most have an identifiable opening, heart, and closing (Stigler & Hiebert, 1999).

In order to acknowledge and accommodate the wide range of lessons that occur in practice, we identified four phases to study. Although other alternative approaches of lesson structure are possible, variations of the four phases we consider herein are observed widely (see, for example, Stigler & Hiebert, 1999, pp. 76-83); this structure is also endorsed by certain Standards-based curricula in the U.S.A., such as the *Connected Mathematics Project* (Martin et al., 2012).

The first phase, lesson planning, precedes the steps identified above and refers to important decisions taken about activities or tasks that will be used in the lesson (Yinger, 1980). Although planning can be for a year or a unit of work, our focus is on the planning done for one lesson.

The second phase is launching a task. This corresponds generally to the preparation and presentation stages of the Herbartian approach (Ivie, 2007). First, students are prepared for the lesson content by reviewing their prior related knowledge, from the day before or from earlier work. Presentation of the new lesson follows. We specifically envisage this as the activation of prior knowledge and the presentation or launch of a mathematical task for students to work on. This phase corresponds to Stigler and Hiebert’s (1999) lesson opening.

The third phase is student autonomous work. Herbart considered this the phase in which students integrate new ideas into clusters to consolidate their understanding thereof (Ivie, 2007). Lampert (2001) refers to "teaching while students work independently" when students work alone or together, with minimal but strategic teacher intervention. The teacher may use this time to gather material for the fourth and final lesson phase. Phase three corresponds to Stigler and Hiebert's (1999) heart of the lesson.

Phase four is whole-class discussion, which corresponds to the Herbartian stages of generalization and application when students are encouraged to identify general rules and principles and to apply these rules to reinforce their understanding (Ivie, 2007). For Lampert (2001), this is the stage in which students can apply mathematical norms to discuss and reason about their solutions to the task assigned. This phase straddles Stigler and Hiebert's (1999) heart of the lesson and lesson closing.

Although our approach to parsing lessons is grounded in the literature, the phases used are sufficiently flexible to accommodate many different lesson approaches. Thus, we expect to find lessons in which the order of the phases differs to those we describe; we also envision finding phases that are minimally present or absent in given lessons or that may recur within the same lesson.

Identifying Practices and Moves

Teaching moves and practices underlie our analysis of what a teacher does. The idea of moves or teaching moves was used in earlier literature to refer to interactions, specifically verbal actions used by teachers to achieve an objective (e.g. Cooney et al., 1975; Moore, 1979; Smith et al., 1967). Although teacher-student exchanges remain important for teaching moves, in more recent literature some authors conceptualize the idea more broadly as "a unit of teaching that has coherence with respect to a purpose" (Banse et al., 2020; Jacob & Empson, 2016, p. 186). It is in this broader sense that we use the term "move"; although many of the moves we describe imply interaction, some describe other, non-interactive teaching actions or decisions.

Collectively we categorize sets of moves as practices of teaching. Although many conceptions of practice arise in teaching (Lampert, 2010), the one of relevance here refers to the professional actions a teacher does habitually as part of their work, akin to a routine. Such practices may be sourced in professional standards or in the work of respected teachers, and such practices contain sub-components (Lampert, 2010). We define practice as a set of moves grounded in mathematics learning used by a teacher to promote student learning (Grossman, et al, 2018).

What is Entailed in Considering Cognitive Activation and Differentiation as Interwoven?

In this section, we discuss the entailments of working at the nexus of cognitive activation and differentiation for four lesson phases: lesson planning, task launching, student autonomous work, and whole-class discussion (see Table 1 for a summary of these entailments). Engaging in a theoretical analysis, for each phase, we first organize these entailments into practices and moves; we then illustrate them using data from the summer school. Although we present these practices and moves as a viable decomposition of the work entailed when working concurrently on cognitive activation and differentiation, three points are important to clarify. First, the two focal constituent elements—cognitive activation and differentiation—are not present to the same degree in any given practice or move; at times, one may be foregrounded and one backgrounded, as the teacher attempts to maximize learning opportunities for all students. Second, although the level of cognitive activation and differentiation might differ across phases, with student autonomous work lending itself better to differentiation than task launching and whole-class discussion, even in the latter two phases all students can benefit from the interactions occurring during the lesson, albeit in different degrees—a point we further explicate below. Third, despite possible overlaps among either practices or moves within and across phases, we are confident that each practice and move has independent merit for this teaching approach, which justifies its inclusion in the proposed decomposition of practice. To reinforce this point, in Table 1 we provide a rationale for the inclusion of each practice and move.

Table 1

Summary of the Practices and Moves Entailed in Working at the Intersection of Cognitive Activation (CA) and Differentiation (DIF) During Lesson Planning and Enactment

Lesson Phase	Practice Oriented Toward Promoting CA and DIF	Move	Rationale (How is working at the intersection of CA and DIF promoted)
A. Lesson Planning (LP)	1. Selecting and analyzing challenging tasks	(a) Selecting challenging tasks (b) Analyzing selected tasks	Absent a challenging task and an analysis of what constitutes the challenge, it is difficult to promote challenge among students with different capacities and needs in terms of responding to mathematical challenge.
	2. Anticipating different students' mathematical learning needs	Based on students': (a) characteristics (b) prior knowledge (c) possible misconceptions (d) alternative ideas	In order to estimate an optimal level of challenge for different (groups of) students, the teacher needs to identify and then take into consideration different students' (mathematical) learning needs in relation to the lesson goals.
	3. Framing and adapting the task's challenge in response to students' different needs	(a) Examining the task's accessibility to all students	Teacher examines whether the task is accessible to all (diverse) students so that all students are challenged and that no student is overwhelmed or frustrated by the task requirements.
		(b) Adjusting the task complexity with enabling and extending prompts	Teacher anticipates how a task's mathematical challenge for students can be eased or raised as necessary.
		(c) Planning questions to support student productive engagement with the task	Teacher identifies questions of varying difficulty to scaffold all students' access to the task. Questions could be planned for the whole-class, groups of students or individual students.
	4. Anticipating organizational issues	(a) Deciding time to be allotted to different lesson activities and determining the manner in which students will work on the task (e.g., asynchronous work)	With limited lesson time, teacher assesses task challenge relative to student readiness and estimates allocation of time for launching task, autonomously working on task, and discussing task. Teacher also anticipates student capacity for responding to task challenge and decides whether to set intermediate or holistic check-in targets for students or a combination of these.
		(b) Deciding students' organization during the lesson and the classroom setting (e.g., flexible grouping)	Teacher assesses challenge of task and student readiness for it and decides whether students work alone, in pairs or in groups and if applicable, how pairs and groups should be composed.

Lesson Phase	Practice Oriented Toward Promoting CA and DIF	Move	Rationale (How is working at the intersection of CA and DIF promoted)
		(c) Determining how to distribute materials	Teacher anticipates how resources (e.g., base-ten materials) may raise, maintain or lower a task's challenge for some or all students, and consequently decides on universal distribution of materials or making materials accessible to particular groups of students.
B. Task Launching (TL)	1. Presenting the task	(a) Making context accessible to all students	In interacting with students, teacher assesses extent to which context may compound/simplify the mathematical challenge of task for different groups of students and frames the task presentation accordingly.
		(b) Activating student prior knowledge of core mathematical ideas	Teacher interacts with students to activate the knowledge required by different students to engage with the mathematical challenge of the task.
		(c) Clarifying mathematical conditions of task	In interacting with students, teacher builds consensus on the mathematical conditions of the task to ensure that all students can work productively on the task's challenge.
		(d) Clarifying shared mathematical language for completing and discussing task	Teacher is explicit about mathematical language to be used to ensure that challenge is mathematical (and not language-related, for example) for all students.
		(e) Checking cognitive demand of task for specific students and support required by these students	In interacting with students, teacher evaluates inherent mathematical challenge for students based on students' initial reaction to the task.
		(f) Clarifying for students the response required by task	Attempting to scaffold students, teacher decides if the expected format of the response to the task needs to be clarified for some or all students and if so, clarifies it or if not refrains from doing so.
	2. Implementing/adapting and making organizational decisions	Communicating expectations around ways of working on task, including tools to be used and time allocation during the autonomous-work phase	When communicating expectations teacher takes students' different needs for support (e.g., visual aids, manipulative materials, allocation of time) into account and judges how working on the task can best preserve its challenge.
C. Autonomous work (AW)	1. Assessing the complexity-achievement alignment	(a) Purposefully monitoring and interacting with students to gather information on how students respond to the task challenge.	Teacher is evaluating student response to challenge of task with a view to modifying challenge for different (groups of) students.

Lesson Phase	Practice Oriented Toward Promoting CA and DIF	Move	Rationale (How is working at the intersection of CA and DIF promoted)
		(b) Posing questions to gain insights into students' reasoning and understanding of the task	Teacher further interrogates student responses to challenge of task to decide how to address all students' needs.
	2. Determining and implementing next instructional moves	(a) Actively observing students' productive struggle	Teacher assesses if student struggle is productive at different levels (class, group, individual students) or not and responds accordingly.
		(b) Making useful modification(s) to task so that students can productively address the task (e.g., offering guiding questions, suggesting organizers to provide structure, shifting to simpler but challenging task, using tiered activities, simplifying core task, using manipulatives)	In response to teacher's assessment of level of challenge, teacher selects tools to appropriately modify the challenge for different (groups of) students, as necessary. Different types of scaffolds are thus provided for students who might face difficulties with the assigned task.
		(c) Extending the challenge for students who are insufficiently challenged by the task (e.g. prompting connections between representations, prompting work at higher level of sophistication, prompting meta-level reasoning about task)	In response to teacher's assessment of level of challenge, teacher extends challenge for (groups of) students who are insufficiently challenged.
		(d) Modifying classroom setting [e.g. (re)grouping students; moving from group to individual work]	Teacher assesses level of challenge for students and decides to change how some, several, or all students continue to work on the task so that they are productively challenged to a level commensurate with their current needs.
D. Whole-class discussion (WCD)	1. Selecting and sequencing the presentation of student solutions/ideas		Having carefully monitored different students' work on task(s), teacher decides order in which ideas will be shared with classmates to ensure maximum engagement of different students in resolution of task challenge.

Lesson Phase	Practice Oriented Toward Promoting CA and DIF	Move	Rationale (How is working at the intersection of CA and DIF promoted)
	2. Holding students accountable for attending to and understanding classmates' ideas		Teacher pushes students to explain their solutions and others' solutions, aspiring that optimally all students can (strive to) make sense of and evaluate different proposed routes to completing the task.
	3. Eliciting of student reasoning and meaning-making		Teacher presses students to engage in reasoning and meaning-making—thus cognitively activating them—based on their current level and needs.
	4. Using students' incorrect or incomplete solutions as a resource for all students' learning		Teacher interrogates, or encourages students to interrogate, why some solutions do not satisfy conditions of task or how students have taken a misstep so that all students have opportunity to learn from the cognitive challenge of analyzing peers' errors and/or the missing parts of a solution.
	5. Highlighting, synthesizing, and extending important mathematical ideas while interacting with students		Teacher encourages diverse students to synthesize, or teacher synthesizes, ideas presented during whole-class discussion to ensure that students follow, understand, and expand on key mathematical ideas, which is essential to their being cognitively challenged at their own level.
	6. Attending to organizational matters	(a) Establishing norms for sharing mathematical ideas so that they are accessible to and valued by all students	Norms are established so that different student ideas are presented, described, and valued en route to collaboratively resolving mathematical challenge of task. Ideas need to be audible and visible to all students and all students need to develop understanding by confidently sharing their ideas.
		(b) Allocating time for discussing different proposed solutions	Teacher monitors students' engagement with the mathematical challenge of the task and allocates time to maximize mathematical learning based on different students' current level of understanding, engagement, and anticipated time needed for different students to achieve new mathematical insights.

Lesson Planning

Lesson planning is a key contributor to quality teaching (e.g., Nilsson, 2009; Reynolds, 1992). Given its role in teaching, we consider four teacher practices and associated moves that are geared toward materializing a dual focus on cognitive activation and differentiation during lesson planning.

Practices and Moves: Theoretical Analysis

Lesson Planning 1 (LP1). Selecting and Analyzing Challenging Tasks

Selecting challenging mathematical tasks provides the bedrock for cognitively engaging all students. As theoretically argued (Doyle, 1983; Sullivan, 2017; Tekkumru-Kisa et al., 2020) and empirically documented (Kunter et al., 2013; Stein et al., 2007; OECD, 2020), challenging tasks can engage students in cognitively activating work, which, in turn, can promote students' deep learning and reasoning. To determine the suitability of a task, the teacher analyzes how a given task serves the teacher's lesson goals and what makes it challenging. Such analyses provide insights into how the task can be modified to productively engage students of different readiness levels.

Lesson Planning 2 (LP2). Anticipating Different Students' Mathematical Learning Needs

The teacher anticipates different students' mathematical learning needs to ensure that the task selected is suitable for students. This includes considering how student characteristics (e.g., issues of language) might support or impede students' work on the task (Cocking & Mestre, 1988), as well as consideration of students' prior knowledge (Hailikari et al., 2007) and possible student misconceptions or alternative ideas (Doerr, 2006).

Lesson Planning 3 (LP3). Framing and Adapting the Task's Challenge in Response to Students' Different Needs

This practice includes three interrelated moves. The first relates to evaluating if the task is accessible to all students. Toward this end, the teacher might explore whether the task's mathematical features (e.g., representations, mathematical vocabulary, and notation), and non-mathematical features (e.g., contextual and linguistic) prevent certain students from accessing the task (Njagi, 2015; Thomas et al., 2015). The teacher explores whether the task offers multiple entry points (e.g., the task consists of multiple parts or uses different representations or manipulatives), multiple processes (i.e., the task could be solved in different ways) or permits multiple products (i.e., students can use different ways of presenting their work) (cf. Tomlinson, 2014). The second move relates to using enablers and extenders to adjust task complexity (see Charalambous et al., 2022a; Sullivan et al., 2006). The third move pertains to planning questions that can support launching the task or scaffolding students' thinking as they work on it. The teacher identifies questions of varying difficulty to support all students' productive engagement with the task.

Lesson Planning 4 (LP4). Anticipating Organizational Issues

The teacher anticipates organizational matters that might impinge on the unfolding of the task and the ways in which students work on it. This includes considerations regarding the time to be allotted to different lesson activities and the manner in which students will work on the task [e.g., asynchronous work: students working on different parts of the task at the same time (Rogers, 2007)] so that all students have the opportunity to work productively on the task or part of it, at their own pace; the organization of students during the lesson and the classroom setting [e.g., flexible grouping: students are organized in different ways according to their needs and progress on the task (McKeen, 2019; Tomlinson, 2014)]; and how to distribute materials during the lessons (e.g., deciding on the extent to which, and how, different materials or manipulatives will be made available to students, as well as how enablers and extenders will be distributed during the lesson).

Practices and Moves in Context

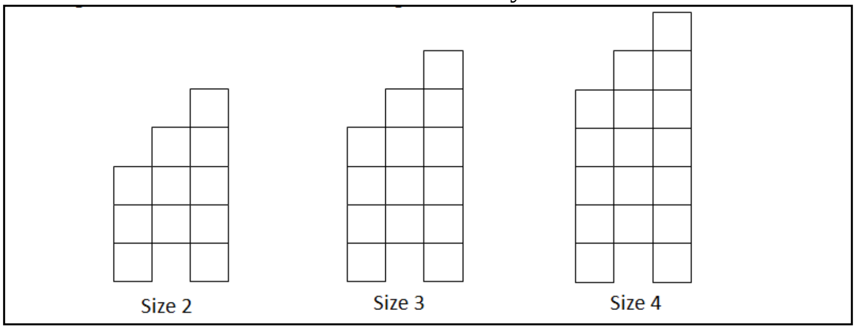
We illustrate how these practices and moves play out during lesson planning by considering Mr. Shea's lesson plan and post-lesson reflection on teaching the *Chairs* task (see Figure 2). As Mr. Shea explains, centered on pattern noticing and algebra, the *Chairs* problem represents a challenging task, since it "requires students to initially continue a given pattern, and then ultimately compose an algebraic expression that enables the calculation of how many tiles are needed to create a specified size of chair". [5] It has the potential to engage students in pattern noticing and generalizing (cf. a "doing-mathematics" task according to Stein et al., 2000). The task lends itself to promoting worthwhile mathematical learning goals, including "recogniz[ing] the mathematical efficiency of creating a rule in order to continue a given pattern" and "translat[ing] the word problem into an algebraic expression that includes a variable and a constant value." Certain aspects render the task challenging: "establishing the constant increase per term in the sequence and continuing the sequence for subsequent sizes", "identifying the constant value and the variable", and "translating the initial rule into an algebraic expression." These need to be considered when preparing a lesson around this task (LP1a, b).

Considering students' prior knowledge needed to work on the task (LP2), Mr. Shea thought that solving this task would require "experience of working with number and non-number patterns; deducing (and recording) the particular rule for a given sequence, and then continuing the sequence to a certain number of terms." He thought some students might struggle with (a) working on the picture of the chairs and might prefer to use more tangible materials; (b) organizing their data to deduce the general rule; or (c) recognizing the significance of chair of Size 1, as a starting point. Anticipating these difficulties and

[5] In quotation marks are the teacher's notes in his lesson plan or the teacher-student exchanges during lesson enactment.

attempting to adjust the task to respond to students' needs (LP3b), he developed enablers to "tweak the challenge for children, without removing it," to ensure that "all would access the task and work productively on it" (post-lesson reflection) (see Figure 3).

Alex uses identical tiles to make different sized chair designs for a school art project. The pictures on the sheet show the first three designs created, size 2, size 3 and size 4. Alex wanted a *rule* that would help work out the number of tiles needed for a chair of any size.



Q 1

- If Alex wanted to create a size 5 chair, what would it look like? Can you draw it or use other materials to represent it? How many tiles would be used?
- Work out the number of tiles needed for the size 6 and size 7 chairs. Explain how you did this.
- Draw or make the size 1 chair. How many tiles did you need?

Q 2

- Do you notice any pattern between the chair size and the number of tiles needed each time? Discuss this pattern with your partner(s).

Q 3

- Alex wanted to create a size 20 chair. Talk with your partner(s) about a *rule* that would help Alex work out the number of tiles needed for this chair.
- Would this rule work for the previous chair sizes?
- If yes, write out this *rule* in words.
- Discuss if it would work for a chair of any size.

Q 4

- Could you re-write this rule using symbols/letters?

Q 5

- Use the *rule* to calculate the number of tiles needed for a "size 50" or a "size 100" chair.

Figure 2. The *Chairs* task.

The first enabler would help students organize their answers in a table, thereby facilitating their attempt to figure out the general rule. For students facing more difficulties, he would provide more scaffolds (second enabler) by decomposing the pattern into the number of tiles in Size-1 chair and then helping them see the constant addition of three tiles each time. The third enabler was designed to help students present the generalization algebraically, something envisaged to puzzle several students.

Task Enabler 1

In order to see a pattern between the chair size and the number of tiles needed each time, it may be useful to organize this information into a table.

Complete the following table using the information you have gathered to date.

Chair Size	**Number of Cards/Tiles Needed**
1	
2	**11**
3	
4	
5	
6	
7	
8	
:	
:	

In your completed table do you notice any connection between the chair sizes and the number of tiles needed?

Task Enabler 2

Chair Size (*S*)	**Number of Tiles Needed (*T*)**	**Explainer**
1	**8**	After drawing the 'size 1' chair, I counted the number of square cards needed. This original count gave me the number **8**.
2	**11**	Original count (8) + 3
3	**14**	Original count (8) + 3 + 3
4	**17**	Original count (8) + 3 + 3 + 3
5	**:**	**:**
6	**:**	**:**
7	**:**	**:**
8	**:**	**:**
:	**:**	**:**
:	**:**	**:**

If *T* represents the 'Number of Tiles Needed', and *S* represents the 'Chair size', then the *rule* that would help work out the number of square cards needed would need to start with T = 8 + ………

Task Enabler 3

Could you re-write this *rule* using symbols/letters?

The following sentence may help:

If *T* represents the "number of tiles needed" and S represents the "chair size", then the rule that would help work out the number of tile cards needed would need to start with T = 8 + …..

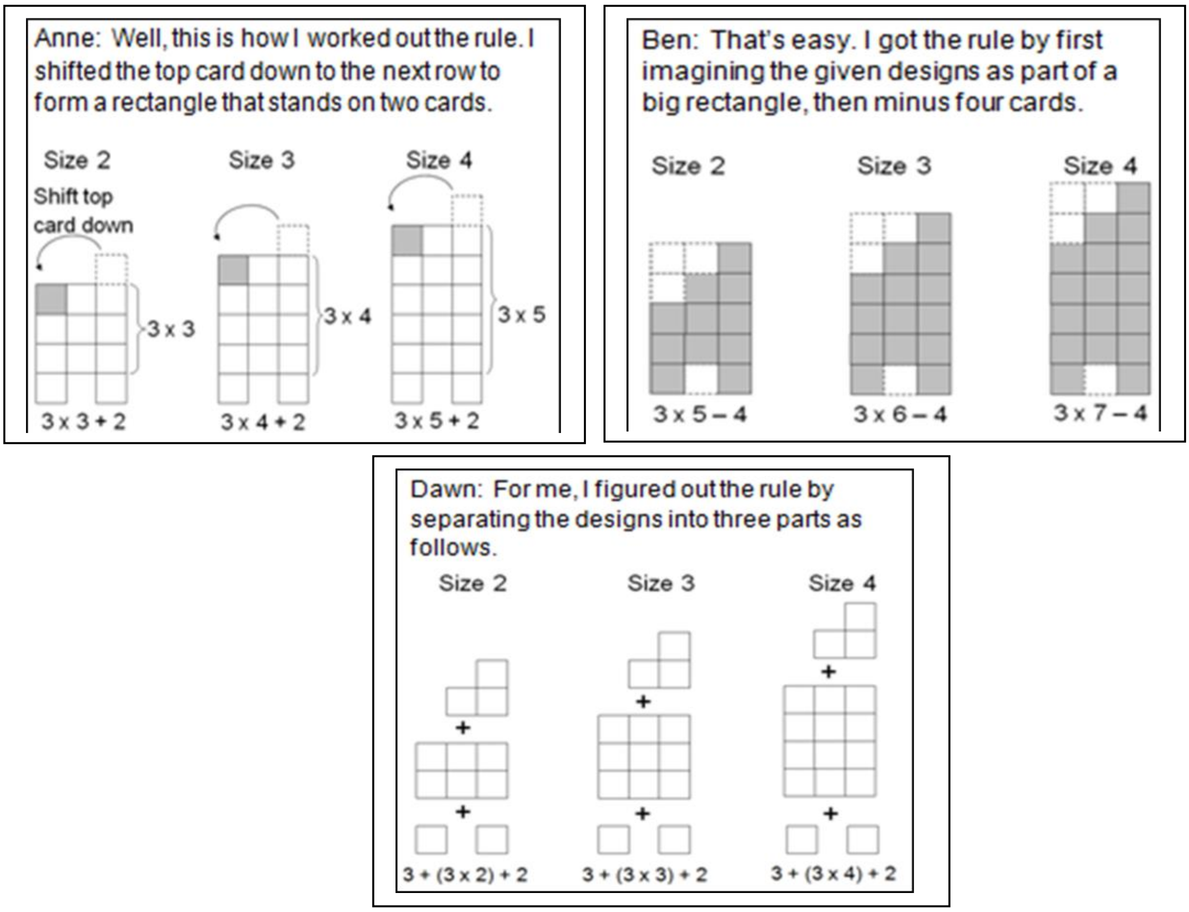
***Figure* 3.** The enablers designed for the *Chairs* task.

Likewise, Mr. Shea anticipated that some students might finish early, even after completing all five task questions. Aspiring to expose these students to multiple ways of thinking about the *Chairs* task, he designed two extenders (see Figure 4) that would extend the challenge (LP3b). Extender 1 presents three different approaches to figuring out the rule. Students assigned this extender would have to understand these approaches and present them symbolically. Extender 2 requires reverse thinking: instead of giving students the size of the chair and asking them to figure out the number of tiles needed to make it, students would be asked to figure out the size of a chair for a given number of tiles.

Further reflecting on presenting the task, Mr. Shea considered how to make its mathematical and non-mathematical content more accessible to all students (LP3a). Although the task includes no difficult terms (except for "pattern" and "rule"), Mr. Shea thought of initially asking students to read it silently, then have one student read it aloud, and then have other students paraphrase it. To ensure that all students would start productively working on the first question, he thought of modifying the task (see Figure 2, modifications in underlined text), giving students the opportunity to present their work in multiple ways. To offer students multiple entry points, he would make square pattern blocks available to students needing them. The fact that the task admitted multiple solution approaches was an additional warrant that it could engage all students in challenging work. Mr. Shea planned which questions to ask (and to whom) (LP3c) so as to press students to "reach up a little bit" (post-lesson reflection).

Reflecting on more organizational issues (LP4a, b), he thought carefully about how long to allocate to each lesson phase in order to avoid rushing students and maintain productive engagement. As he remarked, "If you're challenging students, you have to give them time to think […]. If you're giving time, you're allowing children to think at different rates" (post-lesson reflection). Other organizational planning included making manipulatives available so that students could use them if they wished and having enablers and extenders printed as handouts for distribution. The latter would help him have students work on different task parts.

Task Extender 1:
Other than using the table, there are various ways to find the *rule* that would help work out the number of square cards needed. Three friends, Anne, Ben and Dawn all used different methods which are shown below. Spend some time exploring each of these methods.



For each of these methods, can you write a *rule* using symbols/ letters that would help work out the number of square cards needed for a chair of any size?

(a) Are all of these rules the same?
(b) Are they the same as the rule you created previously?
(c) Using any of the methods/ rules, calculate the number of square cards needed for a "size 50" chair.

Task Extender 2
In what chair size would 230 tiles be needed? Explain how you determined this.

***Figure* 4.** The extenders designed for the *Chairs* task.

Mr. Shea's planning illustrates that productively engaging all students in cognitively demanding work begins before stepping into the classroom. Selecting a challenging task is just one of these decisions, followed by a series of other decisions geared toward ensuring that all students can access the task and that steps are taken to adjust the challenge as needed to meet different needs and readiness levels.

Launching a Task

After selecting a task and anticipating how students will respond to it, the launch phase begins. Two practices are evident in this phase: presenting the task to maintain complexity for all students (Jackson et al, 2012), which focuses on considerations related to the mathematical substance of the task, and making organizational decisions, which focuses on the organizational aspects surrounding the launching of a task.

Practices and Moves: Theoretical Analysis

Task Launching 1 (TL1): Presenting a Task

First the teacher makes the task context accessible to all students. Although tasks set in realistic contexts can motivate students, contexts familiar to some students may be unfamiliar to others (Silver et al., 1995) and additional teacher effort is required to ensure fidelity to such tasks and keep the mathematics in the foreground (Stylianides & Stylianides, 2008). Contextual or language features are often not mathematical but unless explicitly addressed during task presentation, may distract some students from solving the task and focusing on the mathematics.

Next the teacher helps activate students' prior knowledge of core mathematical ideas to be applied in the task. Students are reminded of relevant big mathematical ideas (e.g., Charles, 2005). The third move is to ensure that students understand the task's mathematical conditions. The fourth move pertains to the teacher and students agreeing on mathematical language to be used for completing and discussing the task.

Fifth, although challenge and accessibility are criteria for task selection, the teacher needs to monitor students' participation levels to ascertain if the challenge is appropriate for these specific students and to identify customized supports individual students may require when engaging autonomously with the task. Sixth, the teacher decides if the expected format of task response needs to be clarified for some or all students and acts accordingly. All these moves are intended to ensure that students are able to productively engage with the task at some level.

Task Launching 2 (TL2): Implementing/Adapting and Making Organizational Decisions

The second practice involved in launching the task pertains to implementing or adapting decisions taken during lesson planning and making additional decisions in relation to organizational matters. This involves the teacher communicating expectations around ways of working, including what tools are made available to students (Stein et al, 2008). Students may use various tools, including their knowledge of mathematical conventions and virtual and physical manipulative materials. Use of manipulative materials will be influenced by student choice, and physical accessibility, including proximity of the materials (Moyer & Jones, 2004). Students are told how to seek support if required and how much time they will have for working on the task during the next phase (autonomous work, see Baxter & Williams, 2010).[6] We now return to Mr. Shea's launch of the *Chairs* task with fifth-grade students.

Practices and Moves in Context

Although launching the task logically precedes student autonomous work and whole-class discussion, task launching phases were evident throughout Mr. Shea's lesson (see

[6] Although this practice seems to overlap with others in other phases, it is distinct since it pertains to the communication of expectations rather than their planning or the teacher's actions to reinforce these expectations while interacting with students during their autonomous work.

Figure 5).[7] Aspects of task launching appeared across six different segments of the lesson, with an initial long period followed by shorter follow-up periods, since the teacher opted to launch or clarify parts of the task interspersed between periods of student autonomous work.[8]

Lesson Part 1*

Lesson Phase	Task Launching 1	Student Autonomous Work 1	Task Launching 2	Student Autonomous Work 2	Task Launching 3	Student Autonomous Work 3	Task Launching 4	Student Autonomous Work 4	Task Launching 5 into Whole Class Discussion 1
Duration (mm:ss)	05:13	02:29	00:13	03:24	01:41	01:20	00:11	02:32	31:48

Lesson Part 2*

Lesson Phase	Task Launching 6: Question 3	Student Autonomous Work 5	Task Launching 7	Student Autonomous Work 6	Whole class discussion 2
Duration (mm:ss)	05:05	14:51	01:04	06:39	18:01

***Figure* 5.** Analysis by phase (task launching, student autonomous work or whole class discussion) of the lesson under consideration.

Although each figure in the given task is called a "chair", only in a rough sense do the figures resemble chairs. Thus the context of this task should be accessible to all students (TL1a). Mr. Shea makes no obvious attempt to activate students' prior knowledge of core

[7] The lesson took place over two consecutive days of the summer school (identified as Lesson Part 1 and Lesson Part 2 in Figure 5).
[8] If such clarifications are addressed to the whole class, they count under task (re)launching; if they are addressed to individual students or groups of students, they count under student autonomous work.

mathematical ideas in the task (e.g. around patterns or multiples of 3) but a teacher may do this in lessons prior to the lesson in which such knowledge will be applied (TL1b).

In an attempt to agree with students on information given in the task, what is asked, and on the task conditions (TL1c), Mr. Shea asks students to paraphrase the task and clarifies what the protagonist in the task wants to create; he suggests that students begin the task by either drawing chairs in their notebooks or creating patterns with plastic tiles. Making the task accessible without reducing its challenge through teacher or student actions, demands teacher judgment. When Mr. Shea asks Tim to paraphrase the task, Tim instead begins outlining how he would do the task. Mr. Shea stops him to avoid reducing the mathematical challenge for others. If Tim shares his solution strategy at the outset, some students would be deprived of "discovering" for themselves one way into the task.

Mr. Shea initially asks students to work independently on the first part of Question 1 of the task (to make a "size 5" chair). As students work through the two subsequent parts of Question 1, some notice a pattern. Question 3 of the task challenges students to find a rule to calculate the number of tiles in a larger chair. In light of existing research, it is expected that some students will propose an additive approach (adding a row of 3 for each chair) while others attempt a multiplicative or scaling approach (Clark & Kamii, 1996) by separating the chair into parts that are fixed and parts that change. The terms constant and variable can be linked to the chairs, where the top three tiles and the bottom two tiles are common to every chair and between the top three tiles and bottom two are multiples of three tiles, where the multiple is the chair size. While interacting with the whole-class, Mr. Shea successfully elicits from students the terms constant and variable and negotiates working definitions for these terms. Awareness of these terms reinforces for students each concept and prepares students for analyzing proposed rules for the chair patterns (TL1d).

Mr. Shea estimates students' readiness for completing the task by asking all students to read it silently, then asking one student to read the task aloud and another to paraphrase it. When the student cannot paraphrase it, Mr. Shea asks him to read the task aloud before paraphrasing it. Thus, before students discuss entailments of the task, they get to read the task directly or listen to it being read three times and hear it paraphrased twice. Mr. Shea is familiarizing all students with the task and its requirements, while considering who might require early support when working autonomously (TL1e). Differentiation is evident in the multiple times students read and listen to the task being read, paraphrased or explaining what the task requires; this provides time for all students to become familiar with the task and to think about it at their own level. Listening to classmates read or paraphrase the task likely makes it more accessible to students who find reading difficult or who find it difficult to focus on the task demands; paraphrasing or explaining a task and comparing strategies can also better cognitively activate high-achieving students (Nemeth et al, 2019), by pressing them to articulate their understanding of the task. However, requiring all students to focus on a single large screen may be difficult for students with a visual impairment, with reading difficulties or who need to point to words as they read them. To address this, each student is given an individual copy of the task.

Mr. Shea asks students what the character in the task wants to create; when one student states that a rule is required, Mr. Shea affirms the response. This move establishes common purpose among students before they embark on the task (TL1f).

From an organizational perspective, in order to communicate expectations to students about how to work and what tools are available to them, Mr. Shea asks students to work independently on Question 1 of the task (TL2). Working on this relatively easy question likely helps some low-achieving students engage with and enter the task. Higher-achieving students, who had read the entire task, advance quickly from Question 1 to begin formulating a rule to find any size chair.

Collaboration with a partner is required for Question 2. Writing the rule to find the size of any chair (Question 3) is to be done with the same partner. Partner work is subsequently reinforced when students are asked to present their rules collaboratively in the whole-class discussion phase. In requiring students to work alone or be accountable for presenting the outcome of pair work (as the task itself prompts students to do in Questions 2 and 3, see Figure 2) at different stages, Mr. Shea attempts to maintain the task's challenge for all students (TL2).

Mr. Shea places tiles on the floor in the center of a U-shaped teaching space. Using plastic tiles when working on the task may help some students (TL2). Students can choose to use or not use tiles, giving students agency to decide whether and how to use them in developing their mathematical ideas (Moyer & Jones, 2004).

Later in the lesson, Mr. Shea outlines the algebraic convention of representing unknown quantities using letters. As social-conventional knowledge (Kamii, 2014), using letters in this way is not necessarily something students could deduce without intervention. Introducing this tool does not reduce the challenge but provides greater mathematical power for students to express their mathematical ideas concisely.

In summary, Mr. Shea weaves the task launch throughout the course of the lesson, which maximizes time for students to work on and discuss their solutions. The initial launch lasts five minutes. After working individually for eight minutes, students read Question 2 of the task and subsequently work on this with the person sitting beside them. In another phase of launching the task, Mr. Shea asks students to record the rule in their notebooks when they figure it out. Later in the lesson, Mr. Shea begins launching Question 3 of the task. Oscar reads the part aloud and Maggie paraphrases it. However, before students begin working autonomously in pairs as Mr. Shea intends, Tim intervenes to describe how he has already figured out the number of tiles required to make a size-20 chair. Mr. Shea permits this intervention and thus, a phase intended to launch another part of the task, becomes a whole-class discussion of one solution (analyzed below).

The challenge inherent in the task is maintained throughout the launch phase as it is read and paraphrased by students. At no stage during the launch phase does Mr. Shea or the students appear to compromise the mathematical challenge inherent in the task.

Autonomous Work

Working on challenging tasks with all students during autonomous work involves two main practices aimed at sustaining a learning environment that targets an appropriate level of challenge for students. Although we present these two practices as distinct, they operate in unison forming a coherent whole.

Practices and Moves: Theoretical Analysis

Autonomous Work 1 (AW1): Assessing the Complexity-Achievement Alignment

The first practice entails purposefully monitoring and interacting with students (Lampert, 2001; Nelson, 2001; Schoenfeld, 1998; Shifter, 2001) to gather information on the alignment between the students' level of achievement and the level of task complexity. This, in turn, can help the teacher ensure that the students are working at an appropriate level of challenge and detect instances where the challenge is either high or low for students. The detection of these latter cases can highlight the need for teacher intervention with a view to either lowering or elevating complexity, thus helping the teacher prepare for the whole-class discussion (Brendehur & Frykholm, 2000; Lampert, 2001).

Occasionally, students' interaction with the task is self-evident, in terms of the complexity-achievement alignment and, hence, assessment can be usefully carried out through mere observation of this interaction (AW1a), Baxter & Williams, 2010]. However, frequently mere observation is insufficient and the teacher must elicit more information to supplement his/her understanding of the challenge level encountered by students (Nelson, 2001; Shifter, 2001). In these cases, the teacher can pose open-ended questions (e.g., Could you explain this? What do you mean by that?) so as to unpack students' reasoning (AW1b), Stein et al., 2008].

Autonomous Work 2 (AW2): Determining and Implementing Next Instructional Moves

Information derived from assessing the complexity-achievement alignment facilitates the determination and implementation of subsequent instructional moves aimed at aligning task complexity and student achievement. Quite often decisions are made spontaneously and they automatically lead to the implementation of the corresponding instructional moves in the moment. Depending on the complexity-achievement alignment, the teacher will likely take different moves. That does not negate the teacher's prior lesson planning. However, because of the complexity of working at the intersection of cognitive activation and differentiation, and regardless of how meticulously the teacher plans for a lesson, it remains likely that additional decisions will need to be made spontaneously. Such decisions and those made in advance are often conducive to maintaining the cognitive challenge at intended levels, since they can help reduce (part of) the complexity of teaching in order to minimize the possibility of making decisions that inadvertently reduce the challenge for the class as a whole or for certain groups of students.

In some cases, even though the challenge appears high for certain students, the teacher may refrain from taking immediate action, restricting himself/herself to active surveillance and allowing time for productive struggle. In some cases, this struggle can resolve the difficulty without teacher intervention (AW2a).

However, often the challenge remains persistently and unproductively high. These cases prompt the teacher to modify the task, thus facilitating students' attempt to address the task (AW2b), Sullivan et al., 2015; Tomlinson, 2014]. These modifications could take various forms, including:

- Offering guiding questions to make struggling students aware of their stumbling block and identify productive ways of moving towards the learning goal
- Using organizers intended to provide structure (e.g., tables)
- Taking advantage of tiered activities (Tomlinson, 1999), in which the teacher assumes the flexibility to regulate the pace at which students interact with various parts of the core task and determine specific parts that would be appropriate for different students
- Shifting to tasks dealing with prerequisites to facilitate access to the core task
- Using simpler versions of the core tasks (i.e., enablers) that deviate from the original task in certain aspects, such as the form of representation, the size of the numbers, or the number of steps
- Using manipulatives that enable students to build on specific cases to derive generalized solutions.

For students who seem to perform at a cognitive level significantly higher than the complexity they are presented with by the task at hand, the teacher can extend the challenge (AW2c, Sullivan et al., 2015). In these cases, the teacher may supplement the current task with more demanding sub-tasks that maintain a high challenge (i.e., extenders). Such modifications include:

- Using tasks/prompts requiring students to draw connections between representations to deepen their thinking
- Using tasks/prompts pushing students to work at a higher level of sophistication, e.g., probe the validity of certain ideas (e.g. mathematical rules), draw connections between ideas

- Using tasks/prompts to engage students at meta-level, reflective reasoning about the key ideas involved in the task at hand.

For students who seem to be working at an appropriate level of challenge, the teacher can continue active monitoring of students' work so as to continuously assess the complexity-achievement alignment.

In any case, an additional organizational move could be enacted by the teacher. The teacher may modify the classroom setting (AW2d), by using flexible grouping (see Tomlinson, 2014; Tomlinson et al., 2003). Underlying this move is the premise that no single classroom setting works at all times in terms of sustaining cognitive challenge while concurrently engaging all students. Thus the teacher may find it useful to (re)group students so that students who happen to face similar difficulties are brought to work together with a specific enabler that could scaffold their attempt to overcome those difficulties. In other cases, the learning process could be enhanced by shifting away from the collective towards the individual level, even for a brief interval, provided this could benefit learning.

Practices and Moves in Context

We again step into Mr. Shea's classroom to illustrate how the instructional practices and moves presented above play out during student autonomous work on the *Chairs* task.

At the beginning of Lesson Part 2 (see Figure 5), following the launch of Question 3, the students work in pairs. Mr. Shea circulates in the classroom and carefully observes student work (AW1a). After allowing time to work on the task, Mr. Shea notices that Tim and his partner, although having easily developed the "adding three to the adjacent chair" rule, are still facing difficulties moving from this rule to a general one for any size chair. He decides to draw on previous parts of the task, anchoring the problem to chairs of specific sizes (e.g., sizes 1 - 7) and use them as a frame for the pursuit of a generalized pattern across these instances (AW2a). Building on that, the teacher poses questions like:

Teacher: What size of chair is this [specific instance]?

Tim: Size 3.

Teacher: So, can you relate […] the size, with the number of tiles [pursuit of a generalized pattern]?

At the same time, the teacher is called on by Lucas and Abby who claim that they have identified the general rule for the number of tiles for any size chair. The teacher asks students to write down the rule: "Have you written it down? It's really important to see how it's written". The teacher, then, reads their rule and realizes that they have written something that he does not readily understand. Attempting to elicit more information to supplement his understanding, Mr. Shea asks: "How did you get that?" After students provide an explanation, which is still vague to the teacher, he uses an additional, neutral, probe to gain more insights into their reasoning: "Why are you saying that?" (AW1b). Understanding that the students have come up with a rule that needs some refining, the teacher asks: "What would the rule be for any size chair? Is that always going to be the case? Test your rule with chair sizes 1-7 (from the previous part of the task). If it works, go to the next question" (AW2a).

After some time, the teacher returns to Tim and his partner, and realizes that both students are working in a structured way, developing and completing the table presented in the first part of Figure 6. Attempting to highlight limitations in the students' strategy, Mr. Shea asks the students to use their table to figure out the number of tiles contained in the fiftieth chair (AW2b). Tim says they have not got that far. Immediately, the teacher acknowledges that Tim and his partner have devised a system that goes up to a specific size chair and then draws their attention to the pursuit of a more generic system. The teacher then says, "I'll leave you work on that. I think you have something there that makes sense."

Despite Mr. Shea's attempt to further challenge them, Tim and his partner continue working on the task apparently following the "adding-three" rule (see lower part of Figure 6).

Mr. Shea then notices another pair of students, Daniel and Amaan, who are making little progress on the task at hand. He provides them with Enabler 1 (See Figure 3, AW2b), saying: "This might help you look at it differently. See if the table can help you." After giving them time to work on this enabler, the teacher realizes that they still need additional support. He offers them Enabler 2 (See Figure 3, AW2b).

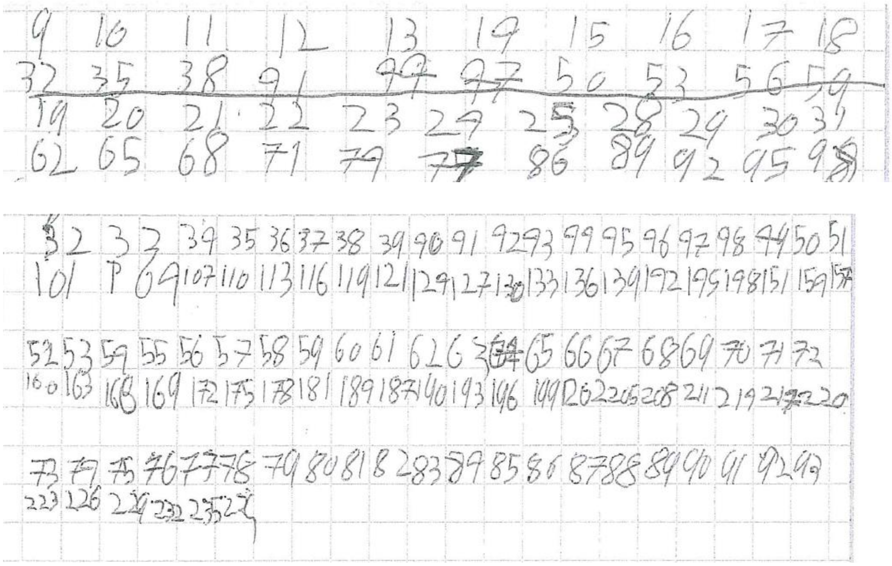


***Figure* 6.** Tim and his partner's autonomous work on Question 3 of the *Chairs* task.

Another student, Lilly, proposes a way of formulating the rule for any size chair; the teacher engages Lilly in the process of formulating coherent arguments in support of her approach, by saying "If you have a way, I want you to convince your partner first of all." After making sure that the student and her partner can explain the rule, the teacher invites them to think about the following: "Have you ever used letters in Math? Try that and explain it to your partner." At a later stage and after Lilly and her partner have successfully written the rule using symbols (see Figure 7), the teacher gives them Extender 1, saying: "This is how some other boys/girls solved the task. I'd like you to read it over and answer the questions" (AW2c). Shortly afterwards and after circulating and observing students' work on the task, he asks Lucas and Abby, who have also finished working on Question 4, to form a larger group with Lilly and her partner and work on Extender 1 (AW2d, flexible grouping).

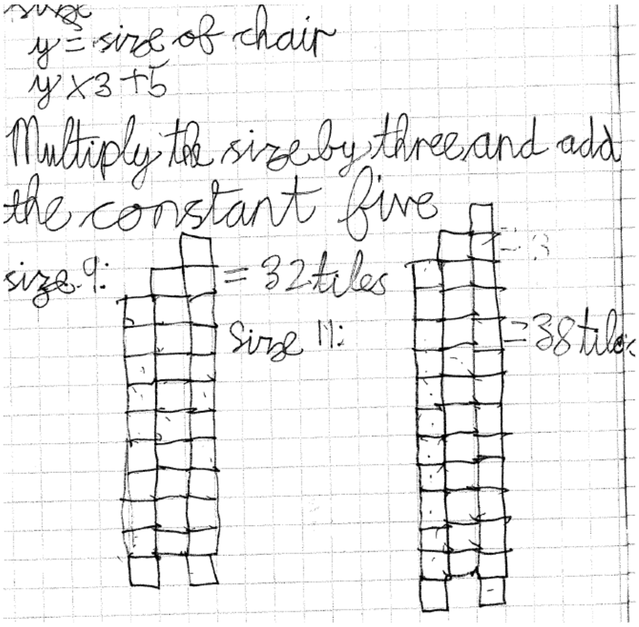


***Figure* 7.** Lilly's work on Question 4 of the Chairs

The above excerpt illustrates Mr. Shea's attempt to concurrently attend to cognitive challenge and differentiation during autonomous work. In doing so, the teacher needs to be *attentive* to departures from what would be deemed a productive complexity-achievement balance while remaining *responsive* through appropriate remedial and extending actions. Undertaking this role, the teacher allowed students to work asynchronously based on their needs and readiness levels.

Whole-Class Discussion

During whole-class discussion the teacher is expected to concurrently address the entire class; hence, this phase imposes different demands than those entailed in the previous phases when trying to differentiate the challenge for different students. Six practices, not assumed to occur in a particular predetermined order—with the exception of the first one—can support the teacher in doing so.

Practices and Moves: Theoretical Analysis

The first whole-class discussion practice (WCD1) pertains to selecting and sequencing the presentation of student ideas in a progression anticipated to support students' understanding (Smith & Stein, 2011). Although such decisions (i.e. who will present, what, and in what order) are largely made in the previous phase, having students' solutions presented in a deliberate sequence can set the ground for keeping different groups of students cognitively active (Stein et al., 2008).

The second practice (WCD2) pertains to holding students accountable for attending to and understanding their classmates' ideas as they are shared in the plenary; this can be done by asking students to revoice or rephrase their classmates' ideas or even compare their classmates' ideas to their own thinking (Chapin et al., 2003).

WCD3 relates to the teachers' eliciting of student reasoning and meaning-making by affording students opportunities to present their thinking, identify and extend patterns, develop generalizations, compare and/or evaluate different approaches, or engage in testing hypotheses or justifying answers—all found to foster deep learning (Baumert et al., 2010; NCTM, 2014; Silver et al., 2005). The next two practices relate to how the teacher capitalizes on students' contributions to cognitively engage their classmates.

Given prior research suggesting that capitalizing on students' partial understanding can support theirs and others' learning (Kazemi & Stipek, 2001; Santagata & Bray, 2016), WCD4 pertains to seizing students' incorrect or incomplete solutions as resources to promote all students' learning. WCD5 concerns the teacher's highlighting, synthesizing, and extending important ideas (while interacting with students)—as opposed to simply eliciting them from students—to make them clear to as many students as possible (Chapin et al., 2003; Smith & Stein, 2011).

WCD6 concerns organizational issues; it includes moves such as establishing and maintaining norms for sharing and discussing mathematical ideas so that they are accessible to all students and valued by all of them (WCD6a)[9] and decisions related to the time allotted to discussing different solutions based on students' needs and current level of understanding (WCD6b). Such moves can set the ground for ensuring that all students have access to key mathematical ideas (NCTM, 2014).

Practices and Moves in Context

Although these practices are not novel, below we illustrate how they can help a teacher keep all students cognitively activated; we do so by stepping into Mr. Shea's classroom when students are sharing solutions to Question 3 of the *Chairs* task.

Noticing that Tim and his partner have worked systematically to address this question—something that was envisioned to support other students' thinking (WCD1)—and cognizant of the fact that Tim needed a push to further articulate his ideas and to realize the need for developing a general rule (WCD6b), Mr. Shea decides to have him present first. However, he first reminds the class of certain norms: closely attending to their classmates' sharing of ideas, comparing them to their own work, and coming up with questions for ideas they might not understand (WCD2). He considers doing so important, both for students like Lucas, Abby, and Lilly who often have more sophisticated ideas and may not pay attention to their classmates' presentation, as well as for students like Ruth, who might not follow others' explanations, thus getting demotivated and going off task (WCD6a).

Tim starts sharing his thinking, while writing on the board:

[9]We see this move as distinct from holding students accountable for attending to and understanding their peers' ideas (WCD2), because whereas the first one refers to creating a classroom environment where every student's contribution is taken seriously and used as a raw material for teaching, the second one pertains to the consequent expectation on students to take seriously their responsibility to understand different ideas, help clarify and interrogate those ideas, while respecting all contributors.

	1	2						
	8	11						

The teacher stops him immediately, asking him to talk louder and to write "a little bit bigger" so that students at the back of the classroom can hear and see (WCD6a). Tim then continues sharing his pattern

1	2	3	4
8	11	14	17

but the teacher stops him again, to clarify what these numbers represent (WCD5) so that the entire class can follow. Several students nominate answers and the teacher presses them to use the correct terminology. Once the class clarifies that the top row corresponds to the size of the chairs, the teacher asks about the bottom row, calling on Ruth, who openly accepts that she is confused. "Being confused is okay," the teacher says, encouraging Ruth to either pose a question to Tim for something she found puzzling or clarify what she has understood up to this point (WCD2 and WCD3). Ruth notices the pattern of "adding three each time," but is unsure as to what the bottom numbers illustrate, which the teacher clarifies while interacting with her and other students. Being asked to follow Tim's rule, the class then continues filling in the table, until Tim makes a mistake:

size	1	2	3	4	5	6	7	8
tiles	8	11	14	17	20	23	26	28

Seizing this teachable moment to engage other students like Lucas and Lilly, who seemed to be off-task, the teacher asks: "What might some problems with this approach be? Why might other approaches that you have thought of work better?" (WCD4). Through the ensuing discussion, the teacher presses students to think of limitations of the "adding three" rule. Doing so surfaces two such limitations (i.e., need to always find the previous term; and when a mistake is made, all subsequent terms are incorrect), but also cognitively activates some students who, despite the teacher's effort, were not paying attention.

The teacher opts to have Lilly's solution shared next, recognizing that hers is more abstract and had it been presented first, it might have confused several students (WCD1). He again reminds students to listen closely to Lilly's solution, thinking how hers is different from Tim's (WCD2 and WCD3). Lilly starts sharing "Basically, you need the size number, say four. Four times three is twelve, plus five is seventeen." Pressing Lilly to further unpack her thinking—thus raising the challenge for her, but also aiming to support other students' understanding—the teacher asks: "What do each of the three numbers stand for?" Lilly explains that five represents the "two tiles at the top and the three tiles at the bottom, which are fixed," but encounters difficulties further articulating her thinking. Mr. Shea invites other students, "Can someone tell us Lilly's rule in their own words?" Although several students contribute ideas, the teacher ensures that the numbers are explicitly connected to the respective diagram. He further challenges students who had solved the problem following Lilly's rule to articulate their thinking: "But how do we know what to multiply three with each time?" Other students are invited to either raise a question or to explain how they understood Lilly's rule (WCD2 and WCD3). Mr. Shea interjects as needed, posing questions to the entire class and ensuring that certain mathematical terms (e.g., constant, variable) are co-constructed and used.

The lesson concludes by the teacher pushing students a step farther, asking them to write in their notebooks at least one similarity or difference between Tim's and Lilly's rules. In doing so, the teacher has all students work at the same task, but differentiates the expected product, since students can come up with different answers and generate as many as they like.

The six foregoing practices do not work in isolation, but rather in unison, when attempting to cognitively activate the class during whole-class discussion. To bring this idea home, it suffices to consider how these practices support the teacher in addressing different students: the more advanced ones, those who perform relatively well, and those who struggle. Take, for example, Lilly, Abby, and Lucas who typically finish their work early, come up with more complex ways of approaching challenging tasks, and often lose interest during whole-class discussion, believing that they might not benefit from others' sharing. The cognitive challenge might be sustained or even elevated for these students when asking them to closely listen to and understand their classmates' solutions and compare them to their own (Lampert, 1990; Silver et al., 2005). Pressing them to unpack their thinking can also challenge them, since prior research has documented the challenge in articulating one's thinking to render it comprehensible (National Research Council, 2000). Students, like Tim, who do relatively well in solving challenging tasks but stick to a particular solution approach can be kept cognitively activated by being pressed to present their solutions and/or being held accountable for understanding others' solutions. Even when the sharing of solutions starts with a solution that involves a misconception or a solution that seems beyond these students' reach, the requirement to try and understand what is being shared or to ask questions about it, can keep these students activated. Students like Ruth, on the other hand, might struggle with not only solving challenging tasks but also following other students' sharing. Creating a classroom culture that allows these students to acknowledge their difficulties and ask for clarifications sets the ground for them to be cognitively engaged. Through the purposeful sequencing of solutions (so that these students can follow) and through the teacher's constant attention to clarifying key mathematical ideas, the teacher can create a classroom learning environment that cognitively challenges these students in their own zone of proximal development (Vygotsky, 1978).

Conclusions

Recognizing the importance of considering teaching dimensions as systems rather than as isolated components (cf. Hiebert &Stigler, in press), in this chapter we focused on two key teaching dimensions, namely cognitive activation and differentiation. Instead of simply juxtaposing them, we considered them as being interwoven and in constant interaction with each other. Aiming to better understand the entailments of teaching at the intersection of cognitive activation and differentiation, we decomposed this intricate and important work into a list of practices and moves, which we grouped into four different interrelated phases: lesson planning, task launching, student autonomous work, and whole-class discussion. We illustrated these practices and moves drawing on the teaching of an experienced teacher and teacher educator during a summer laboratory school whose teaching suggested that working at the intersection of cognitive activation and differentiation *is* possible. This decomposition clearly illustrates the complexities inherent in this work and can have important implications for researchers as well as teachers and teacher educators.

Researchers might find the approach undertaken in this chapter—that is to *conceptually* consider teaching for cognitive activation and differentiation as interwoven—useful when trying to examine how teaching quality relates to student learning. Other approaches for exploring this interplay might pertain to considering the interplay of the two dimensions at the level of operationalization (e.g., developing scales with one dimension at the upper end) or the level of measurement (e.g., using moderation effects). Although there seems to be a broad consensus on raising achievement for all students (cf. Kyriakides et al., 2018; Schleicher, 2014), examining the extent to which teachers can achieve this goal and how this affects student learning is still an open empirical question. The three approaches outlined above provide viable ways of contributing to this vital discussion. Future empirical work that examines how they compare to each other in terms of addressing this question is needed.

Teachers not only need to select and modify challenging tasks, but they have to constantly assess students' needs and readiness levels and adapt their teaching accordingly, ensuring that the level of challenge offered to students is consistent with their zone of proximal development. By offering students who might struggle appropriate levels of scaffolding or cognitive support [to use Kleickmann and colleagues' (2020) words] while also ensuring that high-achieving students are also appropriately challenged, teachers can create challenging learning environments that are customized to their students' needs. This work becomes even more challenging in contemporary classes, given the increased diversity of the student population (OECD, 2012) and the increasing responsibilities and expectations often put before teachers (see, for example, European Commission/EACEA/Eurydice, 2021).

The challenges of such teaching require a dedicated teacher focus at all lesson stages, beginning with the planning. Before the lesson begins, the teacher needs to select and analyze tasks, anticipate multiple student learning needs, relate the challenge to the needs, and anticipate organizational issues that may influence challenge and differentiation. The task needs to be presented and deconstructed in an accessible manner and the organization needs to support differentiation and challenge. When students are working autonomously, the teacher assesses how well aligned are the task challenge and the students' capacities for challenge, while maintaining a view on the subsequent instructional moves. At the whole-class discussion phase, the teacher sequences students' sharing of ideas, holds students accountable for engaging with classmates' ideas, elicits students' reasoning while using errors and misunderstandings as learning resources for the full class, and ensures that key ideas are highlighted, synthesized and extended while attending to the extent to which the classroom organization is conducive to these activities. This demanding work on the teacher's part requires ongoing and dedicated commitment to promoting challenge and differentiating teaching for students.

This decomposition of practice can also have important implications for teacher educators. Despite the challenges outlined above, given the priority placed on ambitious teaching (Cohen, 2011; Lampert et al., 2010) and on issues of access and equity when teaching mathematics (Ball, 2021; NCTM, 2014), teaching at the intersection of cognitive challenge and differentiation offers a worthwhile vision for teachers in contemporary classes. Yet, to realize this vision, teachers need systematic and sustained learning opportunities to experiment with and reflect upon this type of work at different junctures of their career. Starting from initial teacher education, prospective teachers need systematic opportunities for guided experimentation and reflection upon this type of teaching in either approximations of teaching practices that offer the opportunity to "learn to kayak on calm waters" (Grossman et al., 2009, p. 2076) or in real classroom settings while receiving feedback from university tutors. Such opportunities can help future teachers aspire to this type of teaching and the confidence that, albeit hard, it is worth pursuing. In fact, two recent studies (Charalambous et al., 2022a, 2022b) attest to the fact that it is both possible and responsible to offer teachers such opportunities for experimentation and reflection upon practice and that even prospective teachers, to certain degrees, can apply aspects of this teaching in their work during field placement. Yet, such opportunities should not be limited only to initial teacher education: they need to be offered throughout teachers' careers, especially given that practicing teachers might have few concrete images of observing or engaging in this kind of teaching (Lampert et al., 2010).

Although suggesting and empirically corroborating that teaching that simultaneously challenges and differentiates is possible, further work is needed in this area. For instance, in addition to the initial conceptual work around understanding the interplay of cognitive activation and differentiation undertaken in this chapter, further iterations between the practice and theory of such teaching will be required in order to extend the findings presented here, which are necessarily limited in scope. A second direction pertains to a more detailed interrogation of the impact of this teaching on students. Although students demonstrated their engagement in the lesson through their oral sharing of insights and interrogation of ideas and their written responses to lesson tasks and prompts, a deliberate decision was made to focus on the opportunities crafted for student learning; how students actually made use of these opportunities represents another important and fruitful area of exploration (see more on opportunities and use in Fend, 1981 and more recently in Charalambous & Praetorius, 2020).

More than two decades ago, supporting teachers working in disadvantaged areas to engage their students in cognitively challenging work, Silver and colleagues (1996) documented the "revolution of the possible" in reforming middle-grade mathematics teaching. Working against all odds and in a context that doubted the possibility of these students to engage with challenging tasks, let alone learn important mathematics, Silver and colleagues showed that immersing students from disadvantaged backgrounds in challenging work is feasible, responsible, and beneficial, albeit hard work. Such bold visions are still needed today if we are to engage *all* students in cognitively activating work. Yet, visions alone are not particularly helpful if we fail to devise the tools and the strategies that help to materialize them in contemporary classes. It is hoped that the decomposition of practice undertaken in this chapter and the images of teaching provided by Mr. Shea's work contribute, at least to some degree, toward this direction by offering ideas to teacher educators, professional developers and teachers about how they can scaffold prospective and practicing teachers' experimentation with and reflection upon this type of teaching.

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